# Effective linear meson model[*]

D.-U. Jungnickel, C. Wetterich

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany
(e-mail: D.Jungnickel@thphys.uni-heidelberg.de, C.Wetterich@thphys.uni-heidelberg.de)

**Abstract.** The effective action of the linear meson model generates the mesonic $n$–point functions with all quantum effects included. Based on chiral symmetry and a systematic quark mass expansion we derive relations between meson masses and decay constants. The model "predicts" values for $f_\eta$ and $f_{\eta'}$ which are compatible with observation. This involves a large momentum dependent $\eta$–$\eta'$ mixing angle which is different for the on–shell decays of the $\eta$ and the $\eta'$. We also present relations for the masses of the $0^{++}$ octet. The parameters of the linear meson model are computed and related to cubic and quartic couplings among pseudoscalar and scalar mesons. We also discuss extensions for vector and axialvector fields. In a good approximation the exchange of these fields is responsible for the important nonminimal kinetic terms and the $\eta$–$\eta'$ mixing encountered in the linear meson model.

## 1 Introduction

In quantum field theory all effects of quantum fluctuations are incorporated in the effective action $\Gamma$, the generating functional of one–particle irreducible Green functions. From the knowledge of these amplitudes the information about particle masses and decay rates, scattering cross sections, etc. can be extracted in a straightforward manner. In particular, the effective action for the mesons in the lowest mass pseudoscalar octet contains all information on the physics involving only $\pi^\pm$, $\pi^0$, $K^\pm$, $K^0$, $\overline{K}^0$ and $\eta$. We emphasize that all quantum fluctuation effects are already included in the effective action and no further integration over fluctuations has to be performed[1]! Without any further input the effective action can be viewed simply as a coherent description of the information gathered by other means about scattering amplitudes, decay rates, etc. In a very general context it contains already predictive power following from constraints which describe the analyticity properties of the momentum dependence of Green functions or general features like convexity. Furthermore, all exact symmetry relations are automatically embodied in the symmetries of $\Gamma$ or the related Ward identities.

Our aim is to find relations among the $n$–point functions described by $\Gamma$ which go beyond exact symmetry properties and general constraints. This allows to establish relations among physical quantities and to make predictions. (For strong interactions these "predictions" are

more often "postdictions", but they permit an understanding of already measured quantities.) In the case of meson physics the ultimate goal is a computation of the effective mesonic action $\Gamma$ from basic QCD, involving as free parameters only $\alpha_s(M_Z)$ and the current quark masses. We will be concerned here with more modest partial answers which follow from a few simple assumptions about the general properties of the mesonic effective action.

A lot of information can be extracted from approximate chiral $SU_L(3) \times SU_R(3)$ symmetry. In the absence of current quark masses for the up, down and strange quark this is an exact symmetry of the QCD Lagrangian which is believed to be broken spontaneously by the chiral condensate to the vector subgroup $SU_V(3)$. Considering the explicit symmetry breaking by the quark masses $m_u$, $m_d$ and $m_s$ as a small effect and expanding the Green functions in powers of these masses gives rise to the very successful chiral perturbation theory [1,2] in the context of the nonlinear sigma model for the lowest $0^{-+}$ octet. In the present paper this approach is extended to a linear meson or sigma model [3] including also fields for the $\eta'$, the lowest lying scalar $0^{++}$ octet and a scalar singlet. (The latter is often called "$\sigma$ particle", and we use in this work the terms "linear meson model" and "linear sigma model" synonymously.) Together with the $0^{-+}$ octet these fields are combined into a complex $3 \times 3$ matrix $\Phi$ which transforms as a linear $(\overline{\mathbf{3}}, \mathbf{3})$ representation with respect to $SU_L(3) \times SU_R(3)$. The corresponding mesons can be interpreted as quark–antiquark bound states $\overline{q}_L q_R$ [4]. In the absence of quark masses spontaneous chiral symmetry breaking arises through a nonvanishing vacuum expectation value of the scalar singlet described by the real part of $\mathrm{Tr}\,\Phi$. Nonvanishing quark masses also enforce nonzero expectation values of the diagonal part of the scalar octet.

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[1] "Effective actions" are also often used in a different context where only some degrees of freedom are integrated out whereas fluctuations of the remaining degrees of freedom still need to be computed. This is, e.g., the typical setting of chiral perturbation theory and differs from our approach

In this context the possible information from approximate chiral symmetry breaking is twofold: First, there are a few simple linear relations which can be understood easily on the basis of representation theory. A typical example is a Gell-Mann–Okubo type mass relation for the particles in the $0^{++}$ octet. This kind of relation could equally well be understood in the context of an extended nonlinear model including the $\eta'$ and the scalar octet. Beyond this, the linear meson model may imply further constraints for the free parameters remaining in a nonlinear model to a given order in the quark mass expansion. This type of constraint arises typically from nonlinearities in the map from the linear to the nonlinear sigma model and is difficult to classify by representation theory. We observe that the effective action of the nonlinear sigma model for the pseudoscalar octet is completely contained in the effective action for the linear meson model, once restricted to the appropriate degrees of freedom. One may therefore hope to extract some information on those parameters of the nonlinear sigma model which appear in higher orders in the quark mass expansion.

The predictive power of the linear model is greatly enhanced if approximate chiral symmetry is combined with additional assumptions:

(i) The derivative expansion assumes for the inverse propagator that the deviation from a momentum dependence $\sim q^2 + m^2$ is only a small effect. This should hold in a range of $q^2$ in the vicinity of the zero at $q^2 = -m^2$. It amounts to neglecting terms in the effective action which contain more than two derivatives or treating the deviations of the inverse propagators from $q^2 + m^2$ as small corrections in a systematic way. For a determination of masses, mixing angles or decay widths only Green functions with on–shell external momenta are of interest. Hence, it is natural to expand the proper vertices around external momenta corresponding to appropriately chosen "average masses" for each $SU_V(3)$ multiplet. For many observables this leads to a derivative expansion which formally corresponds to an expansion in powers of quark masses. It should be stressed, though, that non–analyticities due to multi–particle thresholds clearly restrict the range of validity of this expansion.

(ii) The expansion in the chiral condensate assumes that the typical mass scale relevant for spontaneous chiral symmetry breaking is small as compared to the typical strong interaction mass scales, as, for instance, the string tension or glueball masses. The discussion of the relevant scales is somewhat subtle (cf. Sect. 10). It is often sufficient to assume that the scale of spontaneous chiral symmetry breaking is not large as compared to other strong interaction scales. In the present paper we do not exploit explicitly this expansion but rather use it in order to establish reasonable ranges for some parameters. We emphasize in this context that the present paper makes no polynomial expansion of the effective action around $\Phi = 0$. We will instead expand in the difference $\Phi - \langle\Phi\rangle$ with $\langle\Phi\rangle$ the expectation value of $\Phi$ in the presence of spontaneous chiral sym-

metry breaking and equal quark masses. The latter is justified by the observation that a given order in the quark mass expansion only involves a maximal power of $\Phi - \langle\Phi\rangle$. Within the expansion around $\langle\Phi\rangle$ the chiral condensate $\sigma_0 \sim \mathrm{Tr}\,\langle\Phi\rangle$ appears as a free parameter. It should be noted that the validity of a polynomial expansion around $\Phi = 0$ would automatically generate a systematic expansion in powers of $\sigma_0$. It is, however, not necessary for this purpose.

Both, the derivative expansion and the expansion in $\sigma_0$ can also be motivated by the observation that a classical linear sigma model is in the class of renormalizable theories and remains there if it is coupled to quarks. (We neglect here the large–distance nonlocalities in the effective quark interactions reflecting confinement induced by the gluonic degrees of freedom.) Quantum fluctuations have then the tendency to induce a flow of the effective couplings towards the Gaussian fixed point (triviality) in the vicinity of which the derivative and polynomial expansions become valid[2]. Because of nonvanishing meson masses the running extends, however, at most over a range somewhere inbetween the GeV scale below which the mesons form as quark bound states [5] and approximately 100 MeV where the pion mass acts an an infrared cutoff (with graduation because of the rich scalar mass spectrum). Nevertheless, the renormalization effects may be substantial due to the existence of strong effective couplings [6] such that the general form of the effective action may already be influenced by the vicinity of the Gaussian fixed point. We do not expect that the derivative expansion or the expansion in the chiral condensate $\sigma_0$ converge very fast under all circumstances. The associated dimensionless expansion parameters are simply not very small. This also holds for the expansion in the strange quark mass in contrast to an expansion in $m_u$ and $m_d$. We will discuss these issues in detail and find that the "rate of convergence" depends quite significantly on the physical quantity considered. As a general rule, the convergence is much better for the flavored mesons than for the non–flavored ones.

(iii) The leading mixing approximation attributes the dominant deviation from the low order results of the quark mass or the derivative expansion to a mixing of states. A prominent example is the $\eta$–$\eta'$ mixing which indeed turns out to be responsible for the comparatively slow convergence of the straightforward quark mass expansion in this sector. Another important feature in this context is the "partial Higgs effect" which describes the mixing with the $0^{-+}$ states contained in the divergence $\partial_\mu \rho_A^\mu$ of the axialvector fields.

The smallest common denominator of all these considerations and the minimal starting point for any systematic study of the linear meson model assumes an effective action consisting of the most general effective potential for

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[2] Nonlocal quark interactions related to confinement counteract this tendency for very low momentum scales. They influence the mesons only indirectly and are suppressed by a nonvanishing constituent quark mass

$\Phi$ and the most general kinetic term involving two derivatives. By this we mean that all invariants consistent with $SU_L(3) \times SU_R(3)$ symmetry have to be included which contribute to a given order in the quark mass expansion. It is crucial in this respect that the most general kinetic term in the effective Lagrangian is not simply $Z_\varphi \operatorname{Tr} \partial_\mu \Phi^\dagger \partial^\mu \Phi$. There are other important invariants involving two derivatives as, for example, a term

$$\frac{1}{8} X_\varphi^- \Big\{ \operatorname{Tr} \left( \Phi^\dagger \partial_\mu \Phi - \partial_\mu \Phi^\dagger \Phi \right) \left( \Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi \right)$$
$$+ \operatorname{Tr} \left( \Phi \partial_\mu \Phi^\dagger - \partial_\mu \Phi \Phi^\dagger \right) \left( \Phi \partial^\mu \Phi^\dagger - \partial^\mu \Phi \Phi^\dagger \right) \Big\} \quad (1.1)$$

After chiral symmetry breaking this term induces different wave function renormalizations for the pseudoscalar and the scalar octets, a momentum dependent mixing of $\eta$ and $\eta'$ and similar effects which all turn out to be quantitatively important! We emphasize that the symmetry breaking effects in the wave function renormalizations $Z_i$ for the various mesons (which are described by the kinetic terms) are as important for an understanding of the meson mass spectrum as the "unrenormalized mass terms" $\overline{M}_i^2$ (described by the effective potential). With effective inverse propagators $\sim Z_i q^2 + \overline{M}_i^2$ the physical masses are given as $M_i = \overline{M}_i Z_i^{-1/2}$. A study of the mass splitting between the scalar and the pseudoscalar octet involves the effect of chiral symmetry breaking on $\overline{M}_i^2$ and $Z_i$. We therefore investigate the kinetic terms in the same way as the effective potential. This explains most of the differences of our results with earlier investigations [7]–[14] where chiral symmetry breaking in the kinetic terms was neglected.

The present paper is devoted to a systematic study of the effective action of the linear meson model based on the considerations discussed above. For the pseudoscalar sector we take as phenomenological input $M_{\pi^\pm}$, $M_{K^\pm}$, $M_{K^0}$, $M_{\eta'}$, $f_\pi$ and $f_{K^\pm}$. Using this we compute partial decay rates for the $\pi^0$, $\eta$ and $\eta'$ into two photons as parameterized by $f_{\pi^0}$, $f_\eta$ and $f_{\eta'}$ as well as other quantities of interest. The perhaps most striking outcome is that $M_\eta$ as well as the decay constants $f_\eta$ and $f_{\eta'}$ are determined essentially as functions of only one additional parameter, with a rather weak dependence on the other couplings present in the linear sigma model. Fixing this parameter by the measured value $M_\eta = 547.5$ MeV we predict to first order in the quark mass expansion and first order in the derivative expansion

$$\frac{f_{\pi^0}}{f_\pi} \simeq 1.00$$
$$\frac{f_\eta}{f_\pi} \simeq 1.23$$
$$\frac{f_{\eta'}}{f_\pi} \simeq 0.91 \ . \quad (1.2)$$

Taking into account the theoretical uncertainties these results are in satisfactory agreement with the experimental observations $(f_{\pi^0}/f_\pi)^{\exp} = 1.00 \pm 0.04$, $(f_\eta/f_\pi)^{\exp} = 1.06 \pm 0.05$ and $(f_{\eta'}/f_\pi)^{\exp} = 0.81 \pm 0.02$. In view of the lowest order result for vanishing quark masses, $(f_\eta/f_\pi)^{(0)} = \sqrt{3}$, $(f_{\eta'}/f_\pi)^{(0)} = \sqrt{3/8}$ this is rather remarkable. The values of $M_\eta$, $f_\eta$ and $f_{\eta'}$ for different parameters of the model can be found in Sects. 7, 9 and 13. One can get an idea about the "robustness" of the estimate (1.2) from the figures and tables of these sections. We also discuss the masses of the mesons in the lowest lying $0^{++}$ octet (Sect. 11). We find that the scalar partner of the $\eta$ has a typical mass of $(1300 - 1400)$ MeV and should be associated with the resonance $f_0(1300)$ [19]. Large mixing effects with two–kaon or four–quark states are characteristic for the isotriplet $a_0(980)$. The resonance $a_0(980)$ may actually be dominantly a two–kaon state and in this case the model suggests a further isotriplet resonance with a mass around 1300 MeV. It may be identified with the reported resonance $a_0(1320)$ [15].

Four main lines enter our systematic analysis:

**(1)** The relations between the pseudoscalar meson mass differences within a given multiplet and the differences in decay constants $f_K - f_\pi$ or $f_{K^\pm} - f_{K^0}$ involve the couplings between two pseudoscalar octets and one or more scalar octets. Up to a wave function renormalization the differences of decay constants correspond to the expectation values $\langle h \rangle$ of the diagonal fields in the scalar octet. Lowest order mass differences follow from the cubic couplings $\sim \operatorname{Tr}(m^2 h)$ once the expectation value of $h$ is inserted. (Here $m$ denotes the traceless hermitean matrix of pseudoscalar octet fields and $h$ that of the scalar octet.) Similarly higher order corrections arise from quartic couplings as $\operatorname{Tr}(mhmh)$ and so on. The quark mass expansion is closely related to an expansion in powers of the $SU_V(3)$–breaking expectation value $\langle h \rangle$. This mechanism is described in detail in Sects. 2 and 3. As a byproduct of our analysis one also gains information on the cubic and quartic couplings involving pseudoscalars and scalars which are relevant for the decay of a scalar into two pseudoscalars etc.

**(2)** We formulate the quark mass expansion as a power series in the parameters $\Delta_u$, $\Delta_d$, $\Delta_s$ which measure the deviation of the scalar expectation values from their values for zero quark masses. In particular, $\Delta_s - \frac{1}{2}(\Delta_u + \Delta_d) \sim f_K - f_\pi$ corresponds to the $SU_V(3)$–breaking induced by the mass of the strange quark, and $\Delta_u - \Delta_d \sim f_{K^\pm} - f_{K^0}$ measures the amount of isospin breaking. In our analysis we actually never need to determine the current quark masses. Our "predictions" involve directly the relations between meson masses and decay constants. Within the language of a general $SU_V(3)$ symmetric model for $m$, $h$, etc. the essential ingredient is the determination of the cubic and quartic couplings between scalars and pseudoscalars as well as the wave function renormalizations from the couplings of the linear meson model. This is done in Sect. 4.

A systematic expression of the pseudoscalar octet mass splitting to order $\Delta$ needs the identification of those terms of the effective potential which contribute to this order whereas the quark mass corrections to the kinetic term can be neglected. Similarly, an estimate to order $\Delta^2$ (corresponding to second order in the quark mass expansion)

involves the effective potential contribution to order $\Delta^2$, corrections to the kinetic terms to order $\Delta$ and a lowest order estimate of the terms involving four derivatives. We find that the apparent convergence of the expansion in $\Delta$ is quite satisfactory for the flavored pseudoscalars $\pi^\pm$, $K^\pm$, $K^0$, $\overline{K}^0$ and the $\pi^0$. On the other hand, the formal series in $\Delta$ does not converge very well in the $\eta$–$\eta'$ sector if the singlet mass term generated by the chiral anomaly is, as usual, considered as a quantity $\sim \mathcal{O}(1)$. The reason are the relatively large mixing effects which are formally of the order $\Delta$. This is combined with the observation that a zeroth order mass term for the $\eta'$ (without mixing) is considerably smaller than the physical $\eta'$ mass and actually not so much larger than the zeroth order mass of the $\eta$. We wish to stress that the apparent convergence of the $\Delta$–expansion in the $\eta$–$\eta'$ sector improves substantially if one includes systematically all effects to a given order in $\Delta$ for all elements of the $2 \times 2$ matrix which describes the inverse propagator of the $\eta$–$\eta'$ system. After diagonalization this procedure amounts for the mass eigenvalues to a partial resummation of terms which are formally of higher order in $\Delta$. We believe that these convergence properties of the quark mass expansion are quite general: The series converges very well only in situations without large effects from mixing of states. If mixing is important, a good convergence can only be obtained if the $\Delta$–corrections are retained for all elements in the relevant matrix. We encounter a very similar situation if we want to interpret the $a_0(980)$ as the isotriplet member of the scalar octet described by $h$.

For a computation of differences in decay constants like $f_K - f_\pi$ the lowest order in the quark mass expansion requires corrections of order $\Delta$ both for the effective potential and the kinetic terms. This implies that all relations between meson mass differences and decay constants need to lowest nontrivial order also the kinetic terms to order $\Delta$. The only exceptions are the Gell-Mann–Okubo mass relations to linear order in $\Delta$. They do not involve the decay constants and the leading quark mass corrections to the kinetic terms cancel for these relations. All other "predictions" of the model involve the quark mass corrections to the kinetic term. In order to be able to study separately the quark mass expansion and the derivative expansion we have split our systematic exploration of the linear meson model to order $\Delta$ into several parts. Sects. 4 and 5 deal with the $\Delta$–expansion of the effective potential. In Sect. 6 we supplement these considerations by a general discussion of corrections $\sim \Delta$ and higher derivative contributions to the kinetic terms. The quark mass corrections to the kinetic terms are computed within the linear meson model in Sect. 8 and the higher derivative contributions are estimated in Sect. 12.

**(3)** The expansion in powers of $\Delta$ is a self–contained systematic formalism. Nevertheless, large mixing effects sometimes prevent a fast convergence of the series. One of the examples encountered in this work is the relatively large momentum dependent off–diagonal element in the inverse $\eta$–$\eta'$ propagator. In turn, the sign and size of this element can be explained by the mixing of the pseudoscalar

octet $m$ and the singlet $p$ with the states corresponding to the divergence of the axialvector fields $\partial_\mu \rho_A^\mu$. These states have the same quantum numbers and the mixing corresponds to the so called "partial Higgs effect". We have performed in Appendix B the corresponding analysis of the vector and axialvector system coupled to pseudoscalars and scalars and estimated the contribution from the exchange of $\partial_\mu \rho_A^\mu$ to the off–diagonal element in the $\eta$–$\eta'$ inverse propagator. This estimate coincides rather well with the value that leads to realistic numbers for $M_\eta$, $f_\eta$ and $f_{\eta'}$! This is rather encouraging since the large $\mathcal{O}(\Delta)$ effects find now a natural explanation. Quite generally, our results lead to the picture that the $\Delta$–expansion as well as the derivative expansion converge well once a large enough basis of states is included. Integrating out such states, however, can lead to large coefficients in the formal $\Delta$–expansion for the remaining states and therefore to a slow convergence. Keeping the additional states or integrating them out without a further truncation of the series to given order in $\Delta$ is equivalent to a resummation of higher order terms from the point of view of the formal $\Delta$–expansion. This improves the convergence substantially. We therefore have given a general discussion of mixing effects in Appendix C, where we also identify the most prominent mixings relevant in our context.

**(4)** The effective action $\Gamma$ includes all quantum fluctuations. If $\Gamma$ is known no further loop calculations are necessary. Nevertheless, it is often useful to estimate the contributions of fluctuations for certain invariants contained in $\Gamma$. An example are the contributions to higher derivative terms which reflect the deviation of an inverse propagator $G^{-1}(q^2)$ from the leading momentum dependence $Zq^2 + \overline{M}^2$. Recently, a reformulation of perturbation theory was based on effective vertices and propagators instead of the classical ones [16]. Here, this means that the contributions of quantum fluctuations to the difference $G^{-1}(q^2) - (Zq^2 + \overline{M}^2)$ can be computed in terms of $Z$, $\overline{M}^2$ and appropriate effective cubic couplings $\gamma_i$. Since the effective vertices have been determined by our analysis we can give in Sect. 12 a quantitative estimate of the contributions from scalar and pseudoscalar fluctuations to the higher derivative terms. They turn out to be substantially smaller than those arising from mixing effects with other states.

Besides its main purpose of a systematic discussion of meson masses and decay constants within the framework of the linear meson model this work also constitutes the basis for several interesting developments: Recently, the parameters of the two–flavor meson model were estimated from a solution of nonperturbative flow equations [6]. The resulting values for $f_\pi$ and the chiral condensate $\langle \overline{\psi}\psi \rangle$ encourage an extension of this approach to three flavors. In this case the nonvanishing mass of the strange quark plays an important quantitative role and needs to be included. The results of such a future estimate of the parameters of the linear meson model can then be compared with the values infered in the present work from a phenomenological analysis. The present results also give an idea which invariants are important and should be retained in the nec-

essary truncation of the exact nonperturbative flow equation.

A determination of the parameters of the linear sigma model in the zero temperature ground state is the starting point for any analysis of the high temperature behavior of this model. This is relevant for the QCD phase transition in the early universe and for ongoing and future heavy ion collision experiments. Within the linear meson model the temperature dependence can be studied along similar lines as in [17]. On a crude qualitative level one may also use a mean field approximation [7]–[10]. The temperature dependence of meson masses – in particular the vector mesons – may be experimentally observable in heavy ion collisions. We should point out, however, that such a study remains reliable only as long as $2\pi T$ is smaller than the compositeness scale above which a mesonic description of strong interaction physics becomes invalid.

Finally, we have not exploited in this work the information we have gained concerning cubic and quartic meson couplings. Especially for the extension of our model to include vector and axialvector mesons (Appendix B) these couplings determine the decay rates and branching ratios. They therefore contain a whole lot of additional "predictions" that can be compared with experiment. We plan to come back to all these issues in future work.

## 2 Meson masses to linear order

In the limit of vanishing quark masses the chiral symmetry $SU_L(3) \times SU_R(3)$ is spontaneously broken to the vector–like $SU_V(3)$ subgroup. With respect to this symmetry breaking the 18 real fields contained in the complex $(\overline{\mathbf{3}}, \mathbf{3})$ representation decompose into a pseudoscalar octet plus singlet and a scalar octet plus singlet. The pseudoscalar octet is associated with the eight (pseudo–)Goldstone bosons $\pi^\pm$, $\pi^0$, $K^\pm$, $K^0$, $\overline{K}^0$, $\eta$ and the pseudoscalar singlet with the $\eta'$ particle. The scalars comprise the isotriplet $a_0$, the strange scalars $K_0^{*\pm}$, $K_0^{*0}$, $\overline{K}_0^{*0}$ as well as two states with $f_0$ quantum numbers, one of them being the "$\sigma$–particle". Chiral symmetry breaking is induced by a nonvanishing expectation value $\sigma_0$ of the scalar singlet. In presence of current quark masses the $SU_V(3)$ symmetry is explicitly broken. This is reflected by nonzero expectation values of the flavor neutral fields within the scalar octet.

This and the next section are devoted to a general analysis of the effects of $SU_V(3)$–breaking without explicit reference to the linear sigma model and spontaneously broken chiral symmetry. These concepts will only be introduced in Sect. 4. In this way we can understand the new relations emerging from the linear sigma model in the context of a more general setting involving only $SU_V(3)$ symmetry.

The octet of pseudoscalar mesons is described by a hermitian traceless $3 \times 3$ matrix $m$, whereas the scalar octet is similarly denoted by $h$. In addition, we have the pseudoscalar and scalar $SU(3)$ singlets $p$ and $s$. These fields are normalized with standard kinetic terms

$$\mathcal{L}_{\mathrm{kin}} = \frac{1}{4} \operatorname{Tr} \partial^\mu m \partial_\mu m + \frac{1}{4} \operatorname{Tr} \partial^\mu h \partial_\mu h$$
$$+ \frac{1}{2} \partial^\mu p \partial_\mu p + \frac{1}{2} \partial^\mu s \partial_\mu s . \tag{2.1}$$

With $SU(3)$ generators $\lambda_z$ obeying $\operatorname{Tr} \lambda_y \lambda_z = 2\delta_{yz}$ we may write $m = m_z \lambda_z$, $h = h_z \lambda_z$ with $m_z$, $h_z$ real. For vanishing quark masses the pseudoscalar octet corresponds to the massless Goldstone bosons. The most general mass term consistent with $SU(3)$ symmetry, charge conjugation $\mathcal{C}$ and parity[3] $\mathcal{P}$ reads (we will work in Euclidean space time throughout this paper)

$$\mathcal{L}_{m_o} = \frac{1}{4} m_m^2 \operatorname{Tr} m^2 + \frac{1}{4} m_h^2 \operatorname{Tr} h^2$$
$$+ \frac{1}{2} m_p^2 p^2 + \frac{1}{2} m_s^2 (s - s_0)^2 . \tag{2.2}$$

Small nonvanishing quark masses can be described by a linear perturbation or external source for the scalar mesons

$$\mathcal{L}_j = -\frac{1}{2} \operatorname{Tr} j_h h - j_s s \tag{2.3}$$

with $j_h$ diagonal and traceless. Both $j_h$ and $j_s$ are linear in the quark masses with[4]

$$j_h + c_s j_s \sim \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} . \tag{2.4}$$

This leads to a shift in the expectation value of the scalar fields

$$\langle s \rangle = u$$
$$\langle h \rangle = w\lambda_3 - \sqrt{3}v\lambda_8 \tag{2.5}$$

There are direct relations between the parameters $v$ and $w$ and the differences of meson decay constants which are explained in Appendix A

$$\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v = \frac{1}{3}\left(2\Delta_s - \Delta_u - \Delta_d\right)$$
$$= \frac{1}{3}\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)$$
$$\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} w = \left(\Delta_u - \Delta_d\right)$$
$$= \overline{f}_{K^\pm} - \overline{f}_{K^0} . \tag{2.6}$$

Here $\Delta_u$ is proportional to the difference of the $\langle \overline{u}u \rangle$ condensate for nonvanishing and vanishing quark mass and similarly for $\Delta_d$ and $\Delta_s$. The quantities $\overline{f}_i$ are related to the meson decay constants $f_i$ by $SU(3)$ breaking wave function renormalizations which will be discussed in Sect. 6.

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[3] Charge conjugation corresponds to a transposition of both matrices $m$ and $h$ whereas parity amounts to $m \to -m$, $h \to h$, $p \to -p$, $s \to s$

[4] The normalization $c_s$ can be inferred from Sect. 4

Until then we take into account the effects of nonvanishing quark masses only in the lowest order in a derivative expansion. In this approximation the kinetic term is given by (2.1) and we may identify $\overline{f}_i$ with $f_i$. The wave function renormalization constants $Z_m$ and $Z_h$ are induced by different kinetic terms for the pseudoscalar and scalar octets within the linear sigma model and will be explained in Sect. 4. The isospin violating expectation value $w$ is very small and we can use the experimental values $f_\pi \equiv f_{\pi^\pm} = 92.4$ MeV, $f_{K^\pm} = 113$ MeV to estimate

$$\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v = 13.7 \text{ MeV} . \tag{2.7}$$

We will see later, (6.21), that the relation (2.7) is modified by quark mass corrections to the kinetic terms leading to $\overline{f}_i \neq f_i$. As a result the value (2.7) will be shifted to 23.3 MeV.

At this stage $u$ is not yet fixed, since we still have to specify $s_0$ or, equivalently, the meaning of $s = 0$. A possible choice would be that $s = 0$ denotes the minimum of the potential in the absence of quark masses, i.e. $s_0 = 0$. For this choice $m_m^2$ vanishes and $\langle s \rangle = u = \sqrt{2/3}\,(\Delta_u + \Delta_d + \Delta_s)\,(Z_s/Z_m)^{\frac{1}{2}}$. It will, however, prove to be more convenient to choose the origin of $s$ corresponding to the minimum of the singlet field in the presence of quark masses. In this case one has $s_0 = -j_s/m_s^2$ and

$$\langle s \rangle = u = 0 , \quad m_m^2 > 0 \tag{2.8}$$

such that $m_m^2 \sim \mathcal{O}(\Delta)$. We note that more generally the choice of $s_0$ fixes the point around which the potential $U(m, h, p, s)$ is expanded and therefore $m_m^2$ as well as the values of all other parameters in a polynomial expansion of $U$ depend on $s_0$. For the choice $u = 0$ we will use $m_m^2$, $v$ and $w$ instead of the current quark masses $m_u$, $m_d$ and $m_s$ as the parameters characterizing the explicit chiral symmetry breaking. Their relation to the quark masses involves the solution of the field equations for $s$ and $h$ in presence of the sources (2.3). With the choice $u = 0$ our expansion parameter is given by $(Z_m/Z_h)^{1/2}v \simeq \frac{2}{3}(\overline{f}_K - \overline{f}_\pi)$. The quark mass expansion turns into a Taylor expansion in the small parameter $(\overline{f}_K - \overline{f}_\pi)/(\overline{f}_K + \overline{f}_\pi) \simeq 0.1(0.2)$. Here the numbers in brackets include the effects of quark mass corrections to the kinetic terms. Correspondingly, the small parameter for isospin violating effects is $(\overline{f}_{K^\pm} - \overline{f}_{K^0})/(\overline{f}_{K^\pm} + \overline{f}_{K^0}) \simeq -2 \cdot 10^{-3}(-3 \cdot 10^{-3})$ (see below). This implies that linear isospin breaking effects give corrections of the same order of magnitude as quadratic quark mass effects. As a rough estimate one expects that the squared meson masses can be computed to linear order in the quark masses with an accuracy of 10–20 percent. This corresponds to a typical uncertainty for the masses of the pseudoscalar octet of around (30–60) MeV. To quadratic order in $\Delta$ this error is expected to decrease to a few MeV.

We want to investigate the dependence of the pseudoscalar meson masses on the parameters $m_m^2$, $v$ and $w$. We will work in this and the next three sections with a kinetic term of the form (2.1) and discuss effects from

modifications of the kinetic term in Sect. 6. To linear order in the quark masses we need the contributions linear in $m_m^2$, $v$ and $w$. This involves the cubic couplings of two pseudoscalars and $h$ or $s$. Their most general form consistent with $SU(3)$ symmetry, charge conjugation and parity reads

$$\mathcal{L}_3 = \frac{1}{4}\gamma_1 s\,\text{Tr}\,m^2 + \frac{1}{4}\gamma_2\,\text{Tr}\,m^2 h + \frac{1}{2}\gamma_3 p\,\text{Tr}\,mh + \frac{1}{2}\gamma_4 sp^2 . \tag{2.9}$$

We note that because of $\langle s \rangle = 0$ only $m_m^2$ and the couplings $\gamma_2$, $\gamma_3$ contribute to the pseudoscalar mass matrix. We find for the masses of the off–diagonal or flavored mesons $\pi^\pm$, $K^\pm$ and $K^0$, $\overline{K}^0$

$$\begin{aligned} \overline{M}_{\pi^\pm}^2 &= m_m^2 - \gamma_2 v \\ \overline{M}_{K^\pm}^2 &= m_m^2 + \frac{1}{2}\gamma_2\,(v + w) \\ \overline{M}_{K^0}^2 &= m_m^2 + \frac{1}{2}\gamma_2\,(v - w) . \end{aligned} \tag{2.10}$$

Again, the bars indicate that the quantities $\overline{M}_i$ differ from the physical meson masses $M_i$ by $SU(3)$ breaking wave function renormalization effects to be discussed in Sect. 6. As long as we restrict the discussion to the kinetic term (2.1) (Sects. 2–5) we can identify $\overline{M}_i$ with $M_i$. Equation (2.10) allows one to express couplings in terms of meson masses

$$\begin{aligned} m_m^2 &= \frac{1}{3}\left(\overline{M}_{\pi^\pm}^2 + \overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2\right) \\ \gamma_2 w &= \overline{M}_{K^\pm}^2 - \overline{M}_{K^0}^2 \\ \gamma_2 v &= \frac{1}{3}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2\right) \end{aligned} \tag{2.11}$$

and to estimate the isospin violation to leading order in the quark masses

$$\begin{aligned} \overline{f}_{K^0} - \overline{f}_{K^\pm} &= -\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} w \\ &= 3\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v\frac{\overline{M}_{K^0}^2 - \overline{M}_{K^\pm}^2}{\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2} \\ &\simeq 0.47 \text{ MeV} \end{aligned} \tag{2.12}$$

which is in good agreement with the result $f_{K^0} - f_{K^\pm} \simeq 0.45$ MeV from chiral perturbation theory [2]. Here we have used electromagnetically corrected masses $M_{\pi^\pm} = 135.1$ MeV, $M_{K^\pm} = 492.4$ MeV, $M_{K^0} = 497.7$ MeV. Including quark mass corrections to the kinetic terms these values are shifted to $(Z_m/Z_h)^{1/2}w \simeq -0.67$ MeV, (6.23), and $f_{K^0} \simeq 113.28$ MeV, (6.24). We will occasionally also use the isospin means $\overline{f}_K = \frac{1}{2}(\overline{f}_{K^\pm} + \overline{f}_{K^0})$ and $\overline{M}_K^2 = \frac{1}{2}(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2)$.

To obtain the masses of the neutral pseudoscalars $\pi^0$, $\eta$ and $\eta'$ we have to diagonalize the general mass term

$$\begin{aligned} &\frac{1}{2}\left(\overline{M}_3^2 m_3^2 + \overline{M}_8^2 m_8^2 + M_p^2 p^2\right) \\ &\quad + \overline{M}_{38}^2 m_3 m_8 + M_{3p}^2 m_3 p + M_{8p}^2 m_8 p \end{aligned} \tag{2.13}$$

which also involves $m_p^2$ and $\gamma_3$ according to

$$M_p^2 = m_p^2$$
$$M_{8p}^2 = -\sqrt{3}\gamma_3 v$$
$$M_{3p}^2 = \gamma_3 w$$
$$\overline{M}_8^2 = m_m^2 + \gamma_2 v = \frac{1}{3}\left(2\overline{M}_{K^\pm}^2 + 2\overline{M}_{K^0}^2 - \overline{M}_{\pi^\pm}^2\right)$$
$$\overline{M}_3^2 = m_m^2 - \gamma_2 v = \overline{M}_{\pi^\pm}^2$$
$$\overline{M}_{38}^2 = \frac{1}{\sqrt{3}}\gamma_2 w = \frac{1}{\sqrt{3}}\left(\overline{M}_{K^\pm}^2 - \overline{M}_{K^0}^2\right) .$$

$$(2.14)$$

We recover the usual lowest order relations of chiral perturbation theory or the "eightfold way" for $\overline{M}_8^2$, $\overline{M}_3^2$ and $\overline{M}_{38}^2$ [18,2]. We will neglect the isospin violating mixings of $m_3$ which only contribute to order $w^2$ to the mass eigenvalues. Diagonalization of the $(m_8, p)$ sector yields the mass eigenstates

$$\eta = m_8 \cos\theta_p - p \sin\theta_p$$
$$\eta' = p \cos\theta_p + m_8 \sin\theta_p .$$

$$(2.15)$$

The masses of $\eta$ and $\eta'$ and the octet–singlet mixing angle $\theta_p$ are given by

$$M_{\eta'}^2 + M_\eta^2 = M_p^2 + \overline{M}_8^2$$

$$M_{\eta'}^2 - M_\eta^2 = \left[\left(M_p^2 - \overline{M}_8^2\right)^2 + 4M_{8p}^4\right]^{\frac{1}{2}} \qquad (2.16)$$

$$\tan\theta_p = \frac{\overline{M}_8^2 - M_p^2 + \left[\left(\overline{M}_8^2 - M_p^2\right)^2 + 4M_{8p}^4\right]^{\frac{1}{2}}}{2M_{8p}^2} .$$

Contrary to the meson masses the mixing angle $\theta_p$ will receive additional contributions $\sim \mathcal{O}(\Delta)$ from modifications of the kinetic terms discussed in Sect. 6.

It will later be our aim to find relations among the parameters $m_p^2$, $m_m^2$, $\gamma_2$, $\gamma_3$. For the moment we only notice that the couplings $\gamma_2$ and $\gamma_3$ are directly related to the partial decay width of the scalar octet into two mesons belonging to the pseudoscalar octet or singlet. In particular, $\gamma_2$ can be extracted from (2.7) and (2.11) and one finds

$$\left(\frac{Z_h}{Z_m}\right)^{\frac{1}{2}}\gamma_2 \simeq 11040 \text{ MeV} . \qquad (2.17)$$

This value changes to $(Z_h/Z_m)^{1/2}\gamma_2 \simeq 4942$ MeV once quark mass corrections to the kinetic terms are included.

We next turn to the scalar masses. To zeroth order they are given by $m_h^2$ in (2.2). To linear order in $\Delta$ we have to supplement (2.9) by

$$\Delta\mathcal{L}_3 = \frac{1}{4}\gamma_5 s \operatorname{Tr} h^2 + \frac{1}{6}\gamma_6 \operatorname{Tr} h^3 + \frac{1}{3}\gamma_7 s^3 \qquad (2.18)$$

leading to

$$\overline{M}_{a_o^\pm}^2 = m_h^2 - 2\gamma_6 v$$
$$\overline{M}_{K_o^{*\pm}}^2 = m_h^2 + \gamma_6(v + w) \qquad (2.19)$$
$$\overline{M}_{K_o^{*0}}^2 = m_h^2 + \gamma_6(v - w) .$$

In the following we will neglect isospin violation in the scalar sector and use the isospin means $M_{K^*}$ and $M_{a_o}$. The diagonal part of the mass matrix for the flavor neutral scalars reads

$$M_s^2 = m_s^2$$
$$\overline{M}_{a3}^2 = m_h^2 - 2\gamma_6 v \qquad (2.20)$$
$$\overline{M}_{f8}^2 = m_h^2 + 2\gamma_6 v$$

whereas the off–diagonal elements are given by

$$M_{3s}^2 = \gamma_5 w$$
$$M_{8s}^2 = -\sqrt{3}\gamma_5 v \qquad (2.21)$$
$$\overline{M}_{af}^2 = \frac{2}{\sqrt{3}}\gamma_6 w .$$

We note that $\gamma_7$ does not enter these $\mathcal{O}(\Delta)$ expressions for the pseudoscalar meson masses. In particular, we find the interesting relation

$$\overline{M}_{f8}^2 = \frac{1}{3}\left(4\overline{M}_{K_o^*}^2 - \overline{M}_{a_o}^2\right) \qquad (2.22)$$

which is the Gell-Mann–Okubo formula in the scalar sector.

# 3 Pseudoscalar meson masses to quadratic order

Before proceeding to an estimate of the various couplings we will analyze in this section the general structure of the pseudoscalar meson masses to quadratic order in $v$ and $w$, assuming a kinetic term of the form (2.1). We stress, however, that the modifications of the kinetic terms for nonvanishing quark masses are of the same order as the effects discussed in this section. They are postponed until Sect. 6 only for the sake of a simple presentation and for separating clearly different orders in the derivative expansion. Mass corrections quadratic in $v$ and $w$ involve the quartic couplings for two pseudoscalars and two scalars. Their general form is given by

$$\mathcal{L}_4 = \frac{1}{4}\delta_1 s^2 \operatorname{Tr} m^2 + \frac{1}{4}\delta_2 s \operatorname{Tr} m^2 h + \frac{1}{2}\delta_3 sp \operatorname{Tr} mh$$
$$+ \frac{1}{2}\delta_4 s^2 p^2 + \frac{1}{4}\delta_5 p^2 \operatorname{Tr} h^2 + \frac{1}{2}\delta_6 p \operatorname{Tr} mh^2$$
$$+ \frac{1}{8}\delta_7 \operatorname{Tr} h^2 \operatorname{Tr} m^2 + \frac{1}{2}\delta_8 \operatorname{Tr} m^2 h^2 + \frac{1}{2}\delta_9 \operatorname{Tr}(mh)^2$$
$$+ \frac{1}{8}\delta_{10}\left(\operatorname{Tr} mh\right)^2 . \qquad (3.1)$$

We note that the couplings $\delta_1, \ldots, \delta_4$ do not contribute to the pseudoscalar masses. The term $\sim \delta_5$ adds effectively to $m_p^2$ an additional piece $\delta_5(3v^2 + w^2)$ whereas $m_m^2$ is supplemented by $\delta_7(3v^2 + w^2)$. The quartic terms contribute to the flavored meson masses

$$\Delta\overline{M}_{\pi^\pm}^2 = \delta_7(3v^2 + w^2) + 2\delta_8(v^2 + w^2)$$
$$\qquad + 2\delta_9(v^2 - w^2)$$
$$\Delta\overline{M}_{K^\pm}^2 = \delta_7(3v^2 + w^2) + \delta_8(5v^2 - 2vw + w^2) \qquad (3.2)$$
$$\qquad - 4\delta_9(v^2 - vw)$$
$$\Delta\overline{M}_{K^0}^2 = \delta_7(3v^2 + w^2) + \delta_8(5v^2 + 2vw + w^2)$$
$$\qquad - 4\delta_9(v^2 + vw)$$

whereas the mass matrix of the neutral pseudoscalars receives corrections

$$\Delta\overline{M}_3^2 = \delta_7(3v^2 + w^2) + 2(\delta_8 + \delta_9)(v^2 + w^2) + \delta_{10}w^2$$
$$\Delta\overline{M}_8^2 = \delta_7(3v^2 + w^2) + \frac{2}{3}(\delta_8 + \delta_9)(9v^2 + w^2) + 3\delta_{10}v^2$$
$$\Delta M_p^2 = \delta_5(3v^2 + w^2)$$
$$\Delta\overline{M}_{38}^2 = -\frac{1}{\sqrt{3}}(4\delta_8 + 4\delta_9 + 3\delta_{10})vw$$
$$\Delta M_{3p}^2 = -2\delta_6 vw$$
$$\Delta M_{8p}^2 = -\frac{1}{\sqrt{3}}\delta_6(3v^2 - w^2) .$$

(3.3)

The lowest order relations between the off–diagonal and diagonal mesons are only modified by the $\delta_9$ and $\delta_{10}$ terms

$$\overline{M}_3^2 = \overline{M}_{\pi^\pm}^2 + 4\delta_9 w^2 + \delta_{10}w^2$$
$$\overline{M}_8^2 = \frac{1}{3}\left(2\overline{M}_{K^\pm}^2 + 2\overline{M}_{K^0}^2 - \overline{M}_{\pi^\pm}^2\right)$$
$$+ 12\delta_9 v^2 + 3\delta_{10}v^2 .$$

(3.4)

The combination $\delta_9 + \delta_{10}/4$ is therefore particularly interesting for the mass relations.

If we neglect isospin violating contributions of order $w^2$ to the mass eigenvalues and treat the flavored meson masses $M_{\pi^\pm}$, $M_{K^\pm}$, $M_{K^0}$ as three input parameters we can, in principle, predict $M_\eta$ and $M_{\eta'}$ as well as the octet–singlet mixing angle $\theta_p$. This requires information about the coupling $\delta_9$, the off–diagonal mass term

$$M_{8p}^2 = -\sqrt{3}v(\gamma_3 + \delta_6 v)$$

(3.5)

as well as the singlet mass term

$$M_p^2 = m_p^2 + 3\delta_5 v^2 .$$

(3.6)

It will be the aim of the next section to discuss these quantities in the context of a linear sigma model. In addition, we can use the relations (2.10) and (3.2) to obtain information about $v$, $w$ and therefore about the differences of decay constants (2.6). Without isospin violating effects we have at this stage four unknown observables $(M_\eta, M_{\eta'}, \theta_p, f_{K^\pm} - f_\pi)$ for which we want to find relations.

Finally we note that the number of parameters can be reduced by absorbing $\delta_5$ and $\delta_7$ into the definitions of $m_p^2$ and $m_m^2$, respectively. This will be done in Sect. 5 in the framework of the linear $\sigma$–model. For $w = 0$ also $\delta_6$ can be absorbed into $\gamma_3$. The four observables depend on the "couplings" $m_p^2 + 3\delta_5 v^2$, $\gamma_3 + \delta_6 v$, $\delta_8$ and $\delta_9 + \delta_1 0/4$.

# 4 Linear meson model

In this section we will compute the couplings of Sects. 2, 3 in the context of the linear meson model (often also called linear sigma model). The fields $m$, $p$, $h$ and $s$ are all contained in a complex $3 \times 3$ matrix $\Phi$ which transforms as $(\overline{\mathbf{3}}, \mathbf{3})$ with respect to the chiral flavor group $SU_L(3) \times SU_R(3)$

$$\Phi \to \mathcal{U}_R \Phi \mathcal{U}_L^\dagger ; \quad \mathcal{U}_R \in SU_R(3) , \quad \mathcal{U}_L \in SU_L(3)$$

(4.1)

and carries nonvanishing axial charge. Including up to two derivatives the effective action of the linear $\sigma$–model can be written as a sum of a potential and a kinetic term plus a source term

$$\Gamma[\Phi] = \int d^4x \, (U + \mathcal{L}_{\rm kin} + \mathcal{L}_j) .$$

(4.2)

As a consequence of the invariance under $SU_L(3) \times SU_R(3)$ symmetry and the discrete transformations[5] $\mathcal{P}$ ($\Phi \to \Phi^\dagger$) and $\mathcal{C}$ ($\Phi \to \Phi^T$) the potential is a function of the four independent invariants [6]

$$\rho = \mathrm{Tr}\,\Phi^\dagger\Phi$$
$$\tau_2 = \frac{3}{2}\mathrm{Tr}\left(\Phi^\dagger\Phi - \frac{1}{3}\rho\right)^2 = \frac{3}{2}\mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 - \frac{1}{2}\rho^2$$
$$\tau_3 = \mathrm{Tr}\left(\Phi^\dagger\Phi - \frac{1}{3}\rho\right)^3 = \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^3 - \frac{2}{3}\tau_2\rho - \frac{1}{9}\rho^3$$
$$\xi = \det\Phi + \det\Phi^\dagger .$$

(4.3)

With respect to the vector–like $SU_V(3)$ symmetry we may decompose

$$\Phi = \overline{\sigma}_0 + \frac{1}{\sqrt{2}}\left(i\Phi_p + \frac{i}{\sqrt{3}}\chi_p + \Phi_s + \frac{1}{\sqrt{3}}\chi_s\right)$$

(4.4)

with traceless hermitian $3 \times 3$ matrices $\Phi_p$, $\Phi_s$ and real singlets ($\overline{\sigma}_0$ is a real positive constant)

$$\chi_s = \frac{1}{\sqrt{6}}\left[\mathrm{Tr}\left(\Phi + \Phi^\dagger\right) - 6\overline{\sigma}_0\right]$$
$$\chi_p = -\frac{i}{\sqrt{6}}\mathrm{Tr}\left(\Phi - \Phi^\dagger\right) .$$

(4.5)

The kinetic term involving two derivatives consistent with $SU_L(3) \times SU_R(3)$ symmetry, $\mathcal{C}$ and $\mathcal{P}$ reads[6]

$$\mathcal{L}_{\rm kin} = Z_\varphi \mathrm{Tr}\,\partial^\mu\Phi^\dagger\partial_\mu\Phi + \frac{1}{4}Y_\varphi\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}V_\varphi\partial^\mu\xi\partial_\mu\xi$$
$$+ \frac{1}{2}\tilde{V}_\varphi\partial^\mu\omega\partial_\mu\omega$$
$$- \frac{1}{8}X_\varphi^-\Big\{\mathrm{Tr}\left(\Phi^\dagger\partial_\mu\Phi - \partial_\mu\Phi^\dagger\Phi\right)\left(\Phi^\dagger\partial^\mu\Phi - \partial^\mu\Phi^\dagger\Phi\right)$$
$$+ \mathrm{Tr}\left(\Phi\partial_\mu\Phi^\dagger - \partial_\mu\Phi\Phi^\dagger\right)\left(\Phi\partial^\mu\Phi^\dagger - \partial^\mu\Phi\Phi^\dagger\right)\Big\}$$
$$- \frac{1}{8}X_\varphi^+\Big\{\mathrm{Tr}\left(\Phi^\dagger\partial_\mu\Phi + \partial_\mu\Phi^\dagger\Phi\right)\left(\Phi^\dagger\partial^\mu\Phi + \partial^\mu\Phi^\dagger\Phi\right)$$
$$+ \mathrm{Tr}\left(\Phi\partial_\mu\Phi^\dagger + \partial_\mu\Phi\Phi^\dagger\right)\left(\Phi\partial^\mu\Phi^\dagger + \partial^\mu\Phi\Phi^\dagger\right)\Big\}$$
$$- \frac{1}{4}W_\varphi \mathrm{Tr}\Big\{\left(\partial_\mu\Phi^\dagger\Phi\partial^\mu\Phi^\dagger\Phi + \Phi^\dagger\partial_\mu\Phi\Phi^\dagger\partial^\mu\Phi\right)$$
$$\times \left(\Phi^\dagger\Phi - \frac{1}{3}\mathrm{Tr}\left(\Phi^\dagger\Phi\right)\right)\Big\}$$

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[5] More precisely, the transformation $\Phi \to \Phi^\dagger$ corresponds to left–right symmetry which is closely related to the parity reflection $\mathcal{P}$

[6] By partial integration we bring all contributions to the kinetic term into the form where the two derivatives act on different fields. The invariance of the last term follows from $V_{aa'}V_{bb'}V_{cc'}\epsilon_{a'b'c'} = \epsilon_{abc}$ for arbitrary $V \in SU(3)$

$$+\frac{1}{2}U_\varphi \epsilon^{a_1 a_2 a_3}\epsilon^{b_1 b_2 b_3}$$

$$\times\left(\Phi_{a_1 b_1}\partial^\mu\Phi_{a_2 b_2}\partial_\mu\Phi_{a_3 b_3}+\Phi^\dagger_{a_1 b_1}\partial^\mu\Phi^\dagger_{a_2 b_2}\partial_\mu\Phi^\dagger_{a_3 b_3}\right)$$

$$+\dots. \tag{4.6}$$

Here $Z_\varphi$, $V_\varphi$, etc. are functions of the four independent scalar $SU_L(3)\times SU_R(3)$ invariants (4.3) and the dots stand for other independent terms which are not relevant for our purposes, as for example $\partial^\mu\tau_2\partial_\mu\tau_2$. (See Sect. 8 for the precise meaning of this statement.) We note that the additional pseudoscalar invariant

$$\omega = i\left(\det\Phi - \det\Phi^\dagger\right) \tag{4.7}$$

will always appear with even powers, since there is no other parity–odd invariant. Since its square can be expressed in terms of (4.3)

$$\omega^2 + \xi^2 = 4\det(\Phi^\dagger\Phi) = 4\exp\left\{\operatorname{Tr}\ln(\Phi^\dagger\Phi)\right\}. \tag{4.8}$$

it can only appear as an independent quantity in combination with derivatives as in (4.6). Evaluating (4.6) for a configuration $\Phi_0 = \operatorname{diag}(\overline{\sigma}_0)$ the structure of the kinetic term for fluctuations of $\Phi$ around $\Phi_0$ simplifies to the form

$$\mathcal{L}_{\rm kin} = \frac{1}{2}Z_m\operatorname{Tr}\partial^\mu\Phi_p\partial_\mu\Phi_p + \frac{1}{2}Z_h\operatorname{Tr}\partial^\mu\Phi_s\partial_\mu\Phi_s$$

$$+\frac{1}{2}Z_p\partial^\mu\chi_p\partial_\mu\chi_p + \frac{1}{2}Z_s\partial^\mu\chi_s\partial_\mu\chi_s \tag{4.9}$$

where the normalization is adapted such that to lowest order in $\overline{\sigma}_0$ one has $Z_m = Z_h = Z_s = Z_p = Z_\varphi$. The fields $\Phi_p$, $\chi_p$, $\Phi_s$, $\chi_s$ have the same transformation properties with respect to $SU(3)$ and parity as $m$, $p$, $h$, $s$, respectively.

So far $\overline{\sigma}_0$ in (4.4) has not been specified. If we want to use an expansion around $\langle s\rangle = 0$ we should identify it with the value of $\frac{1}{6}\left(\operatorname{Tr}\Phi + \operatorname{Tr}\Phi^\dagger\right)$ at the potential minimum in presence of the quark masses, i.e.

$$\overline{\sigma}_0 = \frac{1}{3}\left(\sigma_u + \sigma_d + \sigma_s\right)Z_m^{-\frac{1}{2}}$$

$$= \frac{1}{6}\left(\overline{f}_\pi + \overline{f}_{K^\pm} + \overline{f}_{K^0}\right)Z_m^{-\frac{1}{2}}. \tag{4.10}$$

This corresponds to our choice $u = 0$ in Sect. 2 and leads to the identification (cf. (2.1))

$$\Phi_p = (2Z_m)^{-\frac{1}{2}}m \,,\ \Phi_s = (2Z_h)^{-\frac{1}{2}}h$$
$$\chi_p = Z_p^{-\frac{1}{2}}p \,,\ \chi_s = Z_s^{-\frac{1}{2}}s\,. \tag{4.11}$$

Corrections to the kinetic terms which are linear in the quark masses involve cubic terms like $\operatorname{Tr}\Phi_s\partial^\mu\Phi_p\partial_\mu\Phi_p$ etc. and will be discussed in Sect. 8.

For the configuration $\Phi_0 = \operatorname{diag}(\overline{\sigma}_0)$ the invariants (4.3) take on the values

$$\begin{aligned}\rho_0 &= 3\overline{\sigma}_0^2\\ \xi_0 &= 2\overline{\sigma}_0^3\\ \tau_2 &= \tau_3 = 0\,.\end{aligned} \tag{4.12}$$

We next decompose the invariants $\rho - \rho_0$, $\tau_2$, $\tau_3$ and $\xi - \xi_0$ into the irreducible representations $\Phi_p$, $\Phi_s$, $\chi_p$ and $\chi_s$. Using occasionally the shorthand notation

$$\overline{\chi}_s = \sqrt{6}\overline{\sigma}_0 + \chi_s \tag{4.13}$$

one finds

$$\rho - \rho_0 = \frac{1}{2}\operatorname{Tr}\Phi_p^2 + \frac{1}{2}\operatorname{Tr}\Phi_s^2 + \frac{1}{2}\chi_p^2 + \frac{1}{2}\chi_s^2$$
$$+\sqrt{6}\overline{\sigma}_0\chi_s \tag{4.14}$$

$$\tau_2 = \frac{1}{2}\overline{\chi}_s^2\operatorname{Tr}\Phi_s^2 + \frac{1}{2}\chi_p^2\operatorname{Tr}\Phi_p^2 + \frac{3}{8}\operatorname{Tr}\Phi_s^4 - \frac{1}{8}\left(\operatorname{Tr}\Phi_s^2\right)^2$$

$$+\frac{3}{8}\operatorname{Tr}\Phi_p^4 - \frac{1}{8}\left(\operatorname{Tr}\Phi_p^2\right)^2 + \frac{3}{2}\operatorname{Tr}\Phi_s^2\Phi_p^2$$

$$-\frac{3}{4}\operatorname{Tr}\left(\Phi_s\Phi_p\right)^2 - \frac{1}{4}\operatorname{Tr}\Phi_s^2\operatorname{Tr}\Phi_p^2 + \overline{\chi}_s\chi_p\operatorname{Tr}\Phi_s\Phi_p$$

$$+\frac{\sqrt{3}}{2}\overline{\chi}_s\operatorname{Tr}\Phi_s^3 + \frac{\sqrt{3}}{2}\overline{\chi}_s\operatorname{Tr}\Phi_s\Phi_p^2$$

$$+\frac{\sqrt{3}}{2}\chi_p\operatorname{Tr}\Phi_p^3 + \frac{\sqrt{3}}{2}\chi_p\operatorname{Tr}\Phi_p\Phi_s^2 \tag{4.15}$$

$$\tau_3 = -\frac{1}{3}\left(\operatorname{Tr}\Phi_s^2 + \operatorname{Tr}\Phi_p^2\right)\tau_2 - \frac{1}{72}\left(\operatorname{Tr}\Phi_s^2 + \operatorname{Tr}\Phi_p^2\right)^3$$

$$+\frac{1}{8}\operatorname{Tr}\left[\Phi_s^2 + \Phi_p^2 + \frac{2}{\sqrt{3}}\left(\overline{\chi}_s\Phi_s + \chi_p\Phi_p\right)\right]^3$$

$$+\frac{1}{4\sqrt{3}}\left[\overline{\chi}_s\operatorname{Tr}\left(\Phi_s^2\Phi_p\Phi_s\Phi_p - \Phi_s^3\Phi_p^2\right)\right.$$

$$+\chi_p\operatorname{Tr}\left(\Phi_p^2\Phi_s\Phi_p\Phi_s - \Phi_p^3\Phi_s^2\right)\right]$$

$$+\frac{3}{8}\operatorname{Tr}\left[\Phi_s^4\Phi_p^2 + \Phi_s^2\Phi_p\Phi_s^2\Phi_p - 2\Phi_s^3\Phi_p\Phi_s\Phi_p\right.$$

$$\left.+\Phi_p^4\Phi_s^2 + \Phi_p^2\Phi_s\Phi_p^2\Phi_s - 2\Phi_p^3\Phi_s\Phi_p\Phi_s\right]. \tag{4.16}$$

Including terms quartic in $m$, $h$, $s$, $p$ the invariant $\tau_3$ only contributes

$$\tau_3^{(4)} = 2\sqrt{2}\overline{\sigma}_0^3\operatorname{Tr}\Phi_s^3 + \overline{\sigma}_0^2\left[3\operatorname{Tr}\left(\Phi_s^4 + \Phi_s^2\Phi_p^2\right)\right.$$

$$-\operatorname{Tr}\Phi_s^2\left(\operatorname{Tr}\Phi_s^2 + \operatorname{Tr}\Phi_p^2\right)$$

$$\left.+2\sqrt{3}\left(\chi_s\operatorname{Tr}\Phi_s^3 + \chi_p\operatorname{Tr}\Phi_s^2\Phi_p\right)\right]. \tag{4.17}$$

Using the fact that for an arbitrary traceless $3\times 3$ matrix T

$$\det\left(\alpha + T\right) = \frac{1}{3}\operatorname{Tr}T^3 - \frac{1}{2}\alpha\operatorname{Tr}T^2 + \alpha^3 \tag{4.18}$$

one also obtains

$$\xi - \xi_0 = \frac{1}{3\sqrt{2}}\operatorname{Tr}\Phi_s^3 - \frac{1}{\sqrt{2}}\operatorname{Tr}\Phi_s\Phi_p^2$$

$$-\frac{1}{2}\overline{\sigma}_0\left(\operatorname{Tr}\Phi_s^2 - \operatorname{Tr}\Phi_p^2\right) - \frac{1}{2\sqrt{6}}\chi_s\left(\operatorname{Tr}\Phi_s^2 - \operatorname{Tr}\Phi_p^2\right)$$

$$+\frac{1}{\sqrt{6}}\chi_p\operatorname{Tr}\Phi_s\Phi_p - \overline{\sigma}_0\chi_p^2 - \frac{1}{\sqrt{6}}\chi_s\chi_p^2$$

$$+\sqrt{6}\overline{\sigma}_0^2\chi_s + \overline{\sigma}_0\chi_s^2 + \frac{1}{3\sqrt{6}}\chi_s^3. \tag{4.19}$$

For the choice (4.10) one has $u = 0$ (c.f. Sect. 2) and we denote $v$ and $w$ collectively by $\Delta$. One observes that

the expectation values of the invariants in the presence of quark masses obey

$$\langle \rho - \rho_0 \rangle \sim \mathcal{O}(\Delta^2) , \ \langle \xi - \xi_0 \rangle \sim \mathcal{O}(\Delta^2)$$
$$\langle \tau_2 \rangle \sim \mathcal{O}(\Delta^2) , \ \langle \tau_3 \rangle \sim \mathcal{O}(\Delta^3) . \quad (4.20)$$

The mass squared matrix for the pseudoscalar mesons is obtained from the potential of the linear $\sigma$–model by taking the second derivatives with respect to the parity–odd representations $\Phi_p$ and $\chi_p$, which we will collectively denote by $\varphi^-$. Since the potential is parity–even, all VEVs of single derivatives of the invariants (4.3) vanish. We note that

$$\left\langle \frac{\partial^2 \rho}{\partial \varphi^- \partial \varphi^-} \right\rangle \sim \mathcal{O}(1) , \ \left\langle \frac{\partial^2 \xi}{\partial \varphi^- \partial \varphi^-} \right\rangle \sim \mathcal{O}(1)$$
$$\left\langle \frac{\partial^2 \tau_2}{\partial \varphi^- \partial \varphi^-} \right\rangle \sim \mathcal{O}(\Delta) , \ \left\langle \frac{\partial^2 \tau_3}{\partial \varphi^- \partial \varphi^-} \right\rangle \sim \mathcal{O}(\Delta^2) . \quad (4.21)$$

The most general $SU_L(3) \times SU_R(3)$ symmetric potential can be expanded in a Taylor series around $\rho = \rho_0$, $\xi = \xi_0$, $\tau_2 = 0$ and $\tau_3 = 0$. We now see that a determination of the pseudoscalar masses up to quadratic order in the quark masses requires

$$U = \overline{m}_g^2 (\rho - \rho_0) - \frac{1}{2} \overline{\nu} [\xi - \xi_0 - \overline{\sigma}_0 (\rho - \rho_0)]$$
$$+ \frac{1}{2} \overline{\lambda}_1 (\rho - \rho_0)^2 + \frac{1}{2} \overline{\lambda}_2 \tau_2 + \frac{1}{2} \overline{\lambda}_3 \tau_3 \quad (4.22)$$
$$+ \frac{1}{2} \overline{\beta}_1 (\rho - \rho_0)(\xi - \xi_0) + \frac{1}{2} \overline{\beta}_2 (\rho - \rho_0) \tau_2 +$$
$$\frac{1}{2} \overline{\beta}_3 (\xi - \xi_0) \tau_2 + \frac{1}{2} \overline{\beta}_4 (\xi - \xi_0)^2 + \dots .$$

Here the potential has been normalized such that it vanishes for the configuration $\Phi = \mathrm{diag}(\overline{\sigma}_0)$. We see that to zeroth order in the quark masses the only contributions to the pseudoscalar mass matrix arise from $\overline{\nu}$ and $\overline{m}_g^2$. To linear order we obtain corrections from $\overline{\lambda}_2$ and $\overline{\nu}$ whereas to quadratic order the other $\overline{\lambda}_i$ and the $\overline{\beta}_i$ enter. We add to the effective action a source term

$$\mathcal{L}_j = -\frac{1}{2} \mathrm{Tr} \left( \Phi^\dagger j + j^\dagger \Phi \right) \quad (4.23)$$

which is linear in the real quark mass matrix $M_q = \mathrm{diag}(m_u, m_d, m_s)$

$$j = j^\dagger = a_q M_q . \quad (4.24)$$

We denote the singlet part of the source by

$$j_s = \frac{1}{\sqrt{6}} Z_s^{-\frac{1}{2}} \mathrm{Tr}\, j \quad (4.25)$$

and require

$$\frac{\partial}{\partial \chi_s} (U + \mathcal{L}_j)|_{\Phi = \overline{\sigma}_0} = 0 . \quad (4.26)$$

Our choice (4.10) for $\overline{\sigma}_0$ therefore implies

$$\sqrt{6} \overline{\sigma}_0 \overline{m}_g^2 = j_s Z_s^{\frac{1}{2}} \quad (4.27)$$

and the mass term $\overline{m}_g^2$ is linear in the quark masses.

Comparing (4.22) with (2.2), (2.9) and (3.1) we can now determine the various couplings of the last two sections in terms of those of the linear sigma model. For the pseudoscalar mass terms of (2.2) we find

$$m_m^2 = \overline{m}_g^2 Z_m^{-1}$$
$$m_p^2 = \left( \overline{m}_g^2 + \frac{3}{2} \overline{\nu} \overline{\sigma}_0 \right) Z_p^{-1} \quad (4.28)$$

The cubic couplings of (2.9) contributing to $\mathcal{O}(\Delta)$ read

$$\gamma_1 = \sqrt{6} \left( \overline{\sigma}_0 \overline{\lambda}_1 - \frac{1}{12} \overline{\nu} + \overline{\sigma}_0^2 \overline{\beta}_1 + \overline{\sigma}_0^3 \overline{\beta}_4 \right) Z_s^{-\frac{1}{2}} Z_m^{-1}$$
$$\gamma_2 = \frac{1}{2} \left( 3 \overline{\sigma}_0 \overline{\lambda}_2 + \overline{\nu} \right) Z_h^{-\frac{1}{2}} Z_m^{-1}$$
$$\gamma_3 = \frac{\sqrt{6}}{2} \left( \overline{\sigma}_0 \overline{\lambda}_2 - \frac{1}{6} \overline{\nu} \right) Z_p^{-\frac{1}{2}} Z_h^{-\frac{1}{2}} Z_m^{-\frac{1}{2}}$$
$$\gamma_4 = \sqrt{6} \left( \overline{\sigma}_0 \overline{\lambda}_1 + \frac{1}{6} \overline{\nu} - \frac{1}{2} \overline{\sigma}_0^2 \overline{\beta}_1 - 2 \overline{\sigma}_0^3 \overline{\beta}_4 \right) Z_s^{-\frac{1}{2}} Z_p^{-1}$$
$$\quad (4.29)$$

whereas for the quartic couplings of (3.1) we obtain

$$\delta_1 = \left( \frac{1}{2} \overline{\lambda}_1 + \frac{5}{4} \overline{\sigma}_0 \overline{\beta}_1 + 2 \overline{\sigma}_0^2 \overline{\beta}_4 \right) Z_s^{-1} Z_m^{-1}$$
$$\delta_2 = \sqrt{\frac{3}{2}} \left( \frac{1}{2} \overline{\lambda}_2 - \overline{\sigma}_0 \overline{\beta}_1 + 3 \overline{\sigma}_0^2 \overline{\beta}_2 + 3 \overline{\sigma}_0^3 \overline{\beta}_3 - 2 \overline{\sigma}_0^2 \overline{\beta}_4 \right)$$
$$\times Z_s^{-\frac{1}{2}} Z_h^{-\frac{1}{2}} Z_m^{-1}$$
$$\delta_3 = \left( \frac{1}{2} \overline{\lambda}_2 + \frac{1}{2} \overline{\sigma}_0 \overline{\beta}_1 + 3 \overline{\sigma}_0^2 \overline{\beta}_2 + 3 \overline{\sigma}_0^3 \overline{\beta}_3 + \overline{\sigma}_0^2 \overline{\beta}_4 \right)$$
$$\times Z_s^{-\frac{1}{2}} Z_p^{-\frac{1}{2}} Z_h^{-\frac{1}{2}} Z_m^{-\frac{1}{2}}$$
$$\delta_4 = \left( \frac{1}{2} \overline{\lambda}_1 - \overline{\sigma}_0 \overline{\beta}_1 - 4 \overline{\sigma}_0^2 \overline{\beta}_4 \right) Z_s^{-1} Z_p^{-1}$$
$$\delta_5 = \left( \frac{1}{2} \overline{\lambda}_1 - \frac{3}{4} \overline{\sigma}_0 \overline{\beta}_1 + \overline{\sigma}_0^2 \overline{\beta}_4 + \frac{3}{2} \overline{\sigma}_0^2 \overline{\beta}_2 - 3 \overline{\sigma}_0^3 \overline{\beta}_3 \right)$$
$$\times Z_p^{-1} Z_h^{-1}$$
$$\delta_6 = \frac{\sqrt{6}}{8} \left( \overline{\lambda}_2 + 4 \overline{\sigma}_0^2 \overline{\lambda}_3 \right) Z_p^{-\frac{1}{2}} Z_h^{-1} Z_m^{-\frac{1}{2}}$$
$$\delta_7 = \left( \frac{1}{2} \overline{\lambda}_1 - \frac{1}{4} \overline{\lambda}_2 - \frac{1}{2} \overline{\sigma}_0^2 \overline{\beta}_4 - \overline{\sigma}_0^2 \overline{\lambda}_3 + \frac{3}{2} \overline{\sigma}_0^2 \overline{\beta}_2 + \frac{3}{2} \overline{\sigma}_0^3 \overline{\beta}_3 \right)$$
$$\times Z_h^{-1} Z_m^{-1}$$
$$\delta_8 = \frac{3}{8} \left( \overline{\lambda}_2 + 2 \overline{\sigma}_0^2 \overline{\lambda}_3 \right) Z_h^{-1} Z_m^{-1}$$
$$\delta_9 = -\frac{3}{16} \overline{\lambda}_2 Z_h^{-1} Z_m^{-1}$$
$$\delta_{10} = 0 . \quad (4.30)$$

We next turn to the scalars for which we wish to relate the couplings $m_s^2$, $m_h^2$, $\gamma_5$, $\gamma_6$ to those of the linear sigma model. For this purpose we note that, contrary to the case of the pseudoscalar mesons, the parity–even scalar meson fields, collectively denoted by $\varphi^+$, may also appear to odd powers in the invariants $\rho$, $\xi$, $\tau_2$, $\tau_3$. We therefore need in

addition to (4.20)

$$\left\langle \frac{\partial \rho}{\partial \varphi^+} \right\rangle \sim \mathcal{O}(1) \,, \quad \left\langle \frac{\partial \tau_2}{\partial \varphi^+} \right\rangle \sim \mathcal{O}(\Delta)$$
$$\left\langle \frac{\partial \xi}{\partial \varphi^+} \right\rangle \sim \mathcal{O}(1) \,, \quad \left\langle \frac{\partial \tau_3}{\partial \varphi^+} \right\rangle \sim \mathcal{O}(\Delta^2) \tag{4.31}$$

and

$$\left\langle \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^+} \right\rangle \sim \mathcal{O}(1) \,, \quad \left\langle \frac{\partial^2 \tau_2}{\partial \varphi^+ \partial \varphi^+} \right\rangle \sim \mathcal{O}(1)$$
$$\left\langle \frac{\partial^2 \xi}{\partial \varphi^+ \partial \varphi^+} \right\rangle \sim \mathcal{O}(1) \,, \quad \left\langle \frac{\partial^2 \tau_3}{\partial \varphi^+ \partial \varphi^+} \right\rangle \sim \mathcal{O}(\Delta) \,. \tag{4.32}$$

Hence, the expansion (4.22) of the potential contains exactly those terms required to obtain the scalar masses to linear order in $\Delta$. Furthermore, we see that to zeroth order in $\Delta$ only $\overline{m}_g^2, \overline{\nu}, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\beta}_1$ and $\overline{\beta}_4$ contribute. We find in particular

$$m_s^2 = \overline{m}_g^2 Z_s^{-1} + 6\overline{\sigma}_0 \left( \overline{\sigma}_0 \overline{\lambda}_1 + \overline{\sigma}_0^2 \overline{\beta}_1 + \overline{\sigma}_0^3 \overline{\beta}_4 - \frac{1}{12} \overline{\nu} \right) Z_s^{-1}$$
$$m_h^2 = \overline{m}_g^2 Z_h^{-1} + \overline{\sigma}_0 \left( 3\overline{\sigma}_0 \overline{\lambda}_2 + \overline{\nu} \right) Z_h^{-1} \tag{4.33}$$

and

$$\gamma_5 = \sqrt{6} \left( \overline{\sigma}_0 \overline{\lambda}_1 + \overline{\sigma}_0 \overline{\lambda}_2 - \overline{\sigma}_0^3 \overline{\beta}_4 \right.$$
$$\left. + \frac{1}{12} \overline{\nu} + 3\overline{\sigma}_0^3 \overline{\beta}_2 + 3\overline{\sigma}_0^4 \overline{\beta}_3 \right) Z_s^{-\frac{1}{2}} Z_h^{-1}$$
$$\gamma_6 = \frac{1}{4} \left( 9\overline{\sigma}_0 \overline{\lambda}_2 - \overline{\nu} + 12\overline{\sigma}_0^3 \overline{\lambda}_3 \right) Z_h^{-\frac{3}{2}}$$
$$\gamma_7 = \sqrt{6} \left( \frac{3}{2} \overline{\sigma}_0 \overline{\lambda}_1 + \frac{9}{4} \overline{\sigma}_0^2 \overline{\beta}_1 + 3\overline{\sigma}_0^3 \overline{\beta}_4 - \frac{1}{12} \overline{\nu} \right) Z_s^{-\frac{3}{2}} \,. \tag{4.34}$$

## 5 Parameters of the linear meson model

In this section we will give a first estimate of the values of the parameters of the linear $\sigma$–model. It is based on an expansion in powers of $\Delta$ to lowest order in the derivative expansion[7]. Comparison of the estimates of this section with those of the following ones will allow us to evaluate the quantitative influence of quark mass corrections to the kinetic terms (which are neglected here). The results of this section can also be compared with earlier work [7]–[14] by setting $Z_h = Z_p = Z_m$. Later we will see, however, that $Z_h/Z_m$ deviates substantially from one. We observe that the couplings $\overline{\lambda}_1, \overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3$ and $\overline{\beta}_4$ influence the meson masses only through $m_s^2$, $\delta_5$ and $\delta_7$ whereas $\overline{\lambda}_3$ appears

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[7] The systematic ordering of the derivative expansion is ambiguous to lowest order since a minimal kinetic term must always be included. For our purpose we consider to lowest order the kinetic term (4.9). The first order comprises the most general terms with up to two derivatives, the second order includes four derivatives and so on

in addition in $\delta_6$ and $\delta_8$. The couplings $\delta_5$ and $\delta_7$ modify the relation between the neutral and flavored pseudoscalar masses, (3.3) and (3.4), through the term $3\delta_5 v^2$ in $M_p^2$ and the term $3\delta_7 v^2$ in $\overline{M}_3^2$, $\overline{M}_8^2$ and the flavored pseudoscalar masses. (We neglect here corrections $\sim w^2$.) A redefinition of couplings

$$\overline{m}_g'^2 = \overline{m}_g^2 + (3\delta_7 + 4\delta_8 - 2\delta_9) \, v^2 Z_m$$
$$\overline{\nu}' = \overline{\nu} + \left[ 2\delta_5 Z_p - \left( 2\delta_7 + \frac{8}{3} \delta_8 - \frac{4}{3} \delta_9 \right) Z_m \right] \frac{v^2}{\overline{\sigma}_0}$$
$$\overline{\lambda}_2' = \overline{\lambda}_2 - \frac{1}{3} \left[ 2\delta_5 Z_p + \left( \delta_7 + \frac{4}{3} \delta_8 - \frac{2}{3} \delta_9 \right) Z_m \right] \frac{v^2}{\overline{\sigma}_0^2} \tag{5.1}$$

absorbs this correction in the lowest order masses $m_m^2$, $m_p^2$, $m_h^2$, whereas the corresponding shifts in the $\gamma$'s only contribute to cubic order in $\Delta$. In terms of these couplings one has

$$\overline{M}_{\pi^\pm}^2 = \overline{m}_g'^2 Z_m^{-1} - \frac{1}{6} \left( 3\overline{\lambda}_2' \overline{\sigma}_0 + \overline{\nu}' \right) Z_m^{-\frac{3}{2}}$$
$$\times \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)$$
$$- \frac{1}{6} \left( \overline{\lambda}_2' + \overline{\lambda}_3 \overline{\sigma}_0^2 \right) Z_m^{-2} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)^2$$
$$\overline{M}_{K^\pm}^2 = \overline{m}_g'^2 Z_m^{-1} + \frac{1}{12} \left( 3\overline{\lambda}_2' \overline{\sigma}_0 + \overline{\nu}' \right) Z_m^{-\frac{3}{2}}$$
$$\times \left( 4\overline{f}_{K^\pm} - 2\overline{f}_{K^0} - 2\overline{f}_\pi \right)$$
$$+ \frac{1}{12} \left( \overline{\lambda}_2' + \overline{\lambda}_3 \overline{\sigma}_0^2 \right) Z_m^{-2} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)$$
$$\times \left( 7\overline{f}_{K^0} - 5\overline{f}_{K^\pm} - 2\overline{f}_\pi \right)$$
$$\overline{M}_{K^0}^2 = \overline{m}_g'^2 Z_m^{-1} + \frac{1}{12} \left( 3\overline{\lambda}_2' \overline{\sigma}_0 + \overline{\nu}' \right) Z_m^{-\frac{3}{2}}$$
$$\times \left( 4\overline{f}_{K^0} - 2\overline{f}_{K^\pm} - 2\overline{f}_\pi \right)$$
$$+ \frac{1}{12} \left( \overline{\lambda}_2' + \overline{\lambda}_3 \overline{\sigma}_0^2 \right) Z_m^{-2} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)$$
$$\times \left( 7\overline{f}_{K^\pm} - 5\overline{f}_{K^0} - 2\overline{f}_\pi \right) \tag{5.2}$$

or

$$\overline{m}_g'^2 = \frac{1}{3} Z_m \left( \overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 + \overline{M}_{\pi^\pm}^2 \right) \tag{5.3}$$

and

$$\overline{M}_8^2 = \frac{1}{3} \left( 2\overline{M}_{K^\pm}^2 + 2\overline{M}_{K^0}^2 - \overline{M}_{\pi^\pm}^2 \right)$$
$$- \frac{1}{4} \overline{\lambda}_2' Z_m^{-2} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)^2$$
$$M_p^2 = \left( \overline{m}_g'^2 + \frac{3}{2} \overline{\nu}' \overline{\sigma}_0 \right) Z_p^{-1}$$
$$M_{8p}^2 = -\sqrt{2} \left( \frac{Z_m}{Z_p} \right)^{\frac{1}{2}} \left[ \frac{1}{3} \left( \overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2 \right) \right.$$
$$- \frac{1}{4} \overline{\nu}' Z_m^{-\frac{3}{2}} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)$$
$$\left. - \frac{1}{8} \overline{\lambda}_2' Z_m^{-2} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)^2 \right] \,. \tag{5.4}$$

We will use (5.2), (5.3) to determine $\overline{m}_g'^2$, $\overline{\lambda}_2'$ and $\overline{\nu}'$ for given decay constants, $M_{\pi^\pm}$, $M_{K^\pm}$ and $M_{K^0}$ (once the wave function renormalizations are known). The parameters $\overline{m}_g'^2$ and $\overline{\nu}'$ are independent of $\overline{\lambda}_3$ whereas the dependence of $\overline{\lambda}_2'$ on $\overline{\lambda}_3$ is linear in $\Delta$. Hence, the unflavored pseudoscalar mass matrix given by (5.4) depends only to cubic or higher order on $\overline{\lambda}_3$ and we will therefore neglect $\overline{\lambda}_3$ in the pseudoscalar sector altogether.

In the following we will make a first attempt to estimate the parameters $\overline{m}_g'^2$, $\overline{\nu}'$ and $\overline{\lambda}_2'$. We present the values to different orders in the quark masses in order to gain some intuition for the convergence of the quark mass expansion for these parameters. We use in this section the simplified kinetic term (4.9). To zeroth order in the quark masses the octet–singlet mixing vanishes and $\overline{m}_g^2 = 0$. This yields the zeroth order relations for the mass of the $\eta'$ and the scalar octet

$$M_{\eta'}^2 = \tfrac{3}{2}\overline{\nu}'\overline{\sigma}_0 Z_p^{-1} = \tfrac{3}{2}\nu\sigma_0\frac{Z_m}{Z_p}$$
$$m_h^2 = 3\overline{\lambda}_2'\overline{\sigma}_0^2 Z_h^{-1} + \tfrac{2}{3}M_{\eta'}^2\frac{Z_p}{Z_h} = 3\lambda_2\sigma_0^2\frac{Z_m}{Z_h} + \tfrac{2}{3}M_{\eta'}^2\frac{Z_p}{Z_h} \ . \tag{5.5}$$

Here we have used for the second identities renormalized couplings according to

$$\sigma_0 = Z_m^{\frac{1}{2}}\overline{\sigma}_0 = \tfrac{1}{6}\left(\overline{f}_\pi + \overline{f}_{K^\pm} + \overline{f}_{K^0}\right) \simeq 53.1 \text{ MeV}$$
$$\nu = Z_m^{-\frac{3}{2}}\overline{\nu}' \ , \quad m_g^2 = Z_m^{-1}\overline{m}_g'^2 \ , \tag{5.6}$$
$$\lambda_1 = Z_m^{-2}\overline{\lambda}_1 \ , \quad \lambda_2 = Z_m^{-2}\overline{\lambda}_2' \ , \quad \lambda_3 = Z_m^{-3}\overline{\lambda}_3$$

etc. This yields the zeroth order estimate

$$\nu^{(0)}\frac{Z_m}{Z_p} \simeq 11500 \text{ MeV} \ . \tag{5.7}$$

A first possibility for an estimate of $\lambda_2$ uses the relation between the masses of the scalar and the pseudoscalar octets. For this purpose we need

$$m_h^2 = \tfrac{1}{3}\left(2\overline{M}_{K_o^*}^2 + \overline{M}_{a_o}^2\right) \ , \tag{5.8}$$

where $M_{K_o^*} \simeq 1430$ MeV. The $a_0$–mesons, however, are not unambiguously identified. Usually it is associated with the well established $a_0(980)$ resonance. On the other hand, the neighboring $f_0(980)$ is often assumed to be an $I = 0$, $K\overline{K}$ bound state or a four quark state. One may therefore as well identify the $a_0(980)$ with the corresponding $I = 1$, $K\overline{K}$ "molecules". The only remaining candidate for the $a_0$–mesons is then the possible $a_0(1320)$ resonance [15]. In this section we will use both possibilities with the notation $M_{a_o^\pm} = 1320(983)$ MeV. We obtain

$$m_h \simeq 1394(1298) \text{ MeV}$$
$$\lambda_2^{(a)}\frac{Z_m}{Z_h} \simeq 156.9(126.5) - 72.0\left(\frac{Z_p}{Z_h} - 1\right) \ . \tag{5.9}$$

In addition to the uncertainty in the identification of the $a_0$–meson the value of $\lambda_2$ depends rather sensitively on

ratios of wave function renormalization constants. For this reason we will compute below the value of $\lambda_2$ from an expression involving only properties of the pseudoscalar mesons.

To linear order in the quark masses the mass eigenvalues of the pseudoscalars are not affected by quark mass corrections to the kinetic term (see Sect. 6). We can therefore identify $\overline{M}_i = M_i$ and obtain

$$m_m^2 = m_g^2 = \tfrac{1}{3}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 + \overline{M}_{\pi^\pm}^2\right)$$
$$\simeq (411.7 \text{ MeV})^2 \tag{5.10}$$

$$\gamma_2 v = \left(\tfrac{3}{2}\lambda_2\sigma_0 + \tfrac{1}{2}\nu\right)\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v$$
$$= \tfrac{1}{3}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2\right)$$
$$\simeq (388.9 \text{ MeV})^2 \ . \tag{5.11}$$

We may fix the parameter $\frac{Z_m}{Z_p}\nu$ by the relation

$$M_{\eta'}^2 = m_p^2 = \tfrac{3}{2}\frac{Z_m}{Z_p}\nu\sigma_0 + \tfrac{1}{3}\frac{Z_m}{Z_p}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 + \overline{M}_{\pi^\pm}^2\right) \tag{5.12}$$

and find

$$\nu^{(1)}\frac{Z_m}{Z_p} \simeq \left(9372 - 2124\left[\frac{Z_m}{Z_p} - 1\right]\right) \text{ MeV} \ . \tag{5.13}$$

The coupling $\lambda_2$ can now be infered from (5.11)

$$\lambda_2^{(b)} = \tfrac{2}{3}\frac{(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2)}{(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi)\sigma_0} - \tfrac{2}{9}\frac{Z_p}{Z_m}\frac{M_{\eta'}^2}{\sigma_0^2}$$
$$+ \tfrac{2}{27}\frac{(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 + \overline{M}_{\pi^\pm}^2)}{\sigma_0^2}$$
$$= 77.93 - 72.03\left(\frac{Z_p}{Z_m} - 1\right) \ . \tag{5.14}$$

We note that this determination of $\lambda_2$ uses the ratio of two quantities which are linear in $\Delta$. The relation between $\overline{f}_i$ and the meson decay constants $f_i$ is strongly influenced by quark mass corrections to the kinetic terms already to leading order in the quark mass expansion. From there we expect sizeable corrections to $\lambda_2$. In addition, an estimate of $\lambda_2$ requires information on $Z_p/Z_m$ and involves differences in larger quantities. As a result, $\lambda_2$ will be poorly determined even once the quark mass corrections to the kinetic terms are included. We will find in Sect. 13 typical values $\lambda_2 \simeq 15 - 30$ even for $Z_p$ equal or somewhat smaller than one.

Concerning the scalar meson masses to linear order in $\Delta$ we observe the relation

$$m_h^2 - \frac{Z_m}{Z_h}m_g^2 = 2\sigma_0\left(\frac{Z_m}{Z_h}\right)^{1/2}\gamma_2 \tag{5.15}$$

which translates with (5.11) into

$$\frac{Z_h}{Z_m} = \frac{2\sigma_0}{m_h^2}\frac{\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2\right)}{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)} + \frac{m_g^2}{m_h^2} \ . \quad (5.16)$$

Inserting experimental values this leads to the ratio

$$\frac{Z_h}{Z_m} \simeq 0.69(0.79) \ . \quad (5.17)$$

These values are changed to $Z_h/Z_m \simeq 0.40 - 0.65$ once quark mass corrections to the kinetic terms are included (see (11.6)). From the mass splitting within the scalar octet we infer

$$\frac{\gamma_6}{\gamma_2} = \frac{3}{2}\frac{Z_m}{Z_h}\left(1 + \frac{12\sigma_0^3\lambda_3 - 4\nu}{9\sigma_0\lambda_2 + 3\nu}\right)$$

$$= \frac{\overline{M}_{K_o^*}^2 - \overline{M}_{a_o}^2}{\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2} \simeq 0.67(2.38) \ . \quad (5.18)$$

This relation can be used to estimate the size of $\lambda_3$.

Having computed the parameters $\nu$, $m_g^2$, $\sigma_0$, $\lambda_2$, $\lambda_3$, $Z_m/Z_h$, $v$ and $w$ from $M_{\eta'}^2$, $M_{K^\pm}^2$, $M_{K^0}^2$, $M_{\pi^\pm}^2$, $M_{K_o^*}^2$, $M_{a_o^\pm}^2$, $f_{K^\pm}$ and $f_\pi$ we can now derive other meson properties. Within the approximation $\overline{M}_i = M_i$, $\overline{f}_i = f_i$ used in this section we discuss here briefly the non–flavored pseudoscalar mesons. For a determination of the pseudoscalar mixing angle $\theta_p$ to $\mathcal{O}(\Delta)$ and the $\eta$ and $\eta'$ masses to $\mathcal{O}(\Delta^2)$ we need the off–diagonal element in the mass matrix for the neutral pseudoscalars to $\mathcal{O}(\Delta)$

$$M_{8p}^2 = -\frac{1}{2\sqrt{2}}\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)\left(2\lambda_2\sigma_0 - \frac{1}{3}\nu\right)\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}$$

$$= -\sqrt{2}\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}\left\{\frac{1}{3}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 - 2\overline{M}_{\pi^\pm}^2\right)\right.$$

$$- \frac{\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi}{\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi}$$

$$\left.\times\left[\frac{Z_p}{Z_m}M_{\eta'}^2 - \frac{1}{3}\left(\overline{M}_{K^\pm}^2 + \overline{M}_{K^0}^2 + \overline{M}_{\pi^\pm}^2\right)\right]\right\}. \ (5.19)$$

We note that for $Z_p \simeq Z_m$ the second term almost cancels the first one. Using (2.17) with $\overline{M}_8^2$ given by (2.14) and $M_p \simeq M_{\eta'} \simeq 957.8$ MeV as an experimental input, one finds $\theta_p = -18.7, -7.2, 0.5$ for $Z_p/Z_m = 0.5, 1.0, 1.5$. We will see in Sect. 6 how quark mass corrections to the kinetic terms modify these relations substantially already to linear order in $\Delta$.

For the $\eta$ mass we get to linear order the Gell-Mann–Okubo relation

$$\left(M_\eta^{(1)}\right)^2 = \overline{M}_8^2 = \frac{1}{3}\left(2\overline{M}_{K^\pm}^2 + 2\overline{M}_{K^0}^2 - \overline{M}_{\pi^\pm}^2\right)$$

$$\simeq (566.3 \text{ MeV})^2 \ . \quad (5.20)$$

We can use (5.19) together with the estimate of $\overline{M}_8^2$ to quadratic order in $\Delta$, (5.4), to compute a mass relation

between $M_\eta^2$ and $M_{\eta'}^2$ to quadratic order in $\Delta$. Inserting $\lambda_2$ from (5.14) and using $M_{\eta'}$ as an input we find

$$M_\eta^{(2)} = 521.2, 535.7, 549.9 \text{ MeV} \quad (5.21)$$

for $Z_p/Z_m = 0.5, 1.0, 1.5$. This is already very close to the experimental value $M_\eta = 547.5$ MeV as long as $Z_p/Z_m$ is not too small.

To summarize this section we find that the approximation of a quark mass independent kinetic term (2.1) or (4.9) gives already a reasonable overall picture of the scalar and pseudoscalar mesons. The most important modifications from the quark mass corrections of the kinetic terms are expected for the value of $\lambda_2$ and the mixing angle $\theta_p$. This will, in turn, influence the estimate of $M_\eta^2$ to second order in $\Delta$ and similarly the relation between $M_{\eta'}^2$ and $M_p^2$.

# 6 Quark mass corrections to kinetic terms

The kinetic term (2.1) obtains corrections for nonvanishing quark masses. Expanding around $\Phi_0 = \text{diag}(\overline{\sigma}_0)$ with $\langle\chi_s\rangle = 0$ these corrections involve only the expectation value of $h$. We are only interested here in the kinetic terms for $m$ and $p$. The most general corrections involving two derivatives and being linear in $v$ and $w$ can then be written in the form

$$\mathcal{L}_{\text{kin}}^{(1)} = \frac{1}{4}\omega_m \text{Tr } h\partial^\mu m\partial_\mu m + \frac{1}{2}\omega_{pm}\partial_\mu p \text{ Tr } h\partial^\mu m \ . \quad (6.1)$$

The term $\sim \omega_m$ leads to different wave function renormalization constants for pions, kaons and $m_8$ according to

$$\begin{aligned}
Z_\pi &= 1 - \omega_m v \\
Z_{K^\pm} &= 1 + \tfrac{1}{2}\omega_m(v + w) \\
Z_{K^0} &= 1 + \tfrac{1}{2}\omega_m(v - w) \\
Z_8 &= 1 + \omega_m v \ .
\end{aligned} \quad (6.2)$$

This implies that the renormalized pion mass $M_{\pi^\pm}$ obeys

$$M_{\pi^\pm}^2 = \overline{M}_{\pi^\pm}^2 Z_\pi^{-1} \quad (6.3)$$

where $\overline{M}_{\pi^\pm}^2$ is the mass computed in the previous sections, e.g.

$$\overline{M}_{\pi^\pm}^2 = m_m^2 - \gamma_2 v + \delta_7(3v^2 + w^2)$$
$$+ 2\delta_8(v^2 + w^2) + 2\delta_9(v^2 - w^2) \ . \quad (6.4)$$

Similar relations hold for $M_{K^\pm}^2$, $M_{K^0}^2$, $M_3^2$ and $M_8^2$ whereas $M_{38}^2 = \overline{M}_{38}^2 Z_\pi^{-\frac{1}{2}}Z_8^{-\frac{1}{2}}$. There is also a mixed term $\sim \omega_m w\partial^\mu \times m_3\partial_\mu m_8$. It gives corrections $\sim w^2$ to the $\pi^0$ and $\eta$ masses and will be neglected here.

One should note that the choice of $Z_\pi$, $Z_{K^\pm}$, $Z_{K^0}$ and $Z_8$ is somewhat arbitrary. It depends on the convention for $Z_m$, since a rescaling of $Z_m$ would result in a rescaling of $Z_\pi$, $Z_{K^\pm}$, $Z_{K^0}$ and $Z_8$. We employ here a convention for $Z_m$ where (cf. (6.2))

$$Z_\pi + Z_{K^\pm} + Z_{K^0} = 1 \ . \quad (6.5)$$

Neglecting corrections $\sim w^2$ we may then use

$$\omega_m v = \frac{1}{3} \left( \frac{Z_{K^\pm} Z_{K^0}}{Z_\pi^2} - 1 \right) . \tag{6.6}$$

The difference between $\overline{M}^2$ and $M^2$ influences the symmetry relations once expressed in terms of physical masses $M^2$. In particular, we observe a modification of the relation (3.4) between $M_8^2$, $M_{\pi^\pm}^2$, $M_{K^\pm}^2$ and $M_{K^0}^2$. Including the terms (6.1) this relation now reads (see (6.9) for the definition of $\overline{f}$)

$$M_8^2 = \frac{1}{3} \left( 2M_{K^\pm}^2 \frac{Z_{K^\pm}}{Z_8} + 2M_{K^0}^2 \frac{Z_{K^0}}{Z_8} - M_{\pi^\pm}^2 \frac{Z_\pi}{Z_8} \right)$$
$$- \frac{1}{4} \lambda_2 \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)^2 \frac{1}{Z_8} . \tag{6.7}$$

In consequence, the corrections due to the modification of the kinetic term influence the pseudoscalar mass eigenvalues to second order in the quark masses. Expanding (6.7) to this order one finds neglecting terms $\sim w^2$

$$M_8^2 = \frac{1}{3} \left( 2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2 \right)$$
$$- \frac{1}{3} \omega_m v \left( M_{K^\pm}^2 + M_{K^0}^2 - 2M_{\pi^\pm}^2 \right)$$
$$- \frac{1}{4} \lambda_2 \Big[ (f_{K^\pm} + f_{K^0} - 2f_\pi)$$
$$- \frac{1}{4} \omega_m v \left( f_{K^\pm} + f_{K^0} + 4f_\pi \right) \Big]^2 . \tag{6.8}$$

Here we have used that the deviation of $Z_\pi$, $Z_{K^\pm}$, etc. from unity also influences the relation between $v$, $w$ and the decay constants $f_\pi$, $f_{K^\pm}$, $f_{K^0}$. The effect of the wave function renormalization on the meson decay constants is discussed in Appendix A and leads to

$$\left( \frac{Z_m}{Z_h} \right)^{\frac{1}{2}} w = Z_{K^\pm}^{-\frac{1}{2}} f_{K^\pm} - Z_{K^0}^{-\frac{1}{2}} f_{K^0} = \overline{f}_{K^\pm} - \overline{f}_{K^0}$$
$$\left( \frac{Z_m}{Z_h} \right)^{\frac{1}{2}} v = \frac{1}{3} \left( Z_{K^\pm}^{-\frac{1}{2}} f_{K^\pm} + Z_{K^0}^{-\frac{1}{2}} f_{K^0} - 2Z_\pi^{-\frac{1}{2}} f_\pi \right) \tag{6.9}$$
$$= \frac{1}{3} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right)$$

with

$$\overline{f}_\pi = Z_\pi^{-\frac{1}{2}} f_\pi \tag{6.10}$$

and similarly for $f_{K^\pm}$, $f_{K^0}$.

In summary, we will denote the physical meson masses and decay constants by $M_i$ and $f_i$. They correspond to a normalization of the fields with inverse propagator $q^2 + M_i^2$ in the vicinity of $q^2 = -M_i^2$. This is also the relevant field normalization for the decay constants — see Appendix A. On the other hand, the quantities $\overline{M}_i$ and $\overline{f}_i$ correspond to a common $SU(3)$ symmetric wave function renormalization for the whole octet. Symmetry relations are therefore most easily expressed in terms of $\overline{M}_i$ and $\overline{f}_i$. For an approximation of the kinetic term to lowest order in the quark masses there is no difference between $M_i$ and $\overline{M}_i$ or $f_i$ and $\overline{f}_i$. This approximation is sufficient to compute the meson masses to linear order in the quark masses,

but not for decay constants and the mixing angle $\theta_p$. In Sects. 2–5 we employed a lowest order approximation to the kinetic term and therefore omitted the distinction between $M$ and $\overline{M}$ or $f$ and $\overline{f}$ for the quantitative estimates. On the other hand, the algebraic relations in the preceding sections are all expressed in terms of $\overline{M}$ and $\overline{f}$ and are therefore not altered by modifications of the kinetic terms. In consequence, the only necessary change for the quantitative estimates involves the relations between $\overline{f}$ and $f$ or $\overline{M}$ and $M$.

We also note that the relations (6.2) use a wave function renormalization which is defined for a common momentum $q_0^2$ for the whole octet. In fact, one may view the derivative expansion as a Taylor expansion of the inverse propagators $G^{-1}(q)$ around some fixed nonvanishing momentum $q_0^2$ rather than an expansion around zero momentum, i.e. an expansion of the type $G_i^{-1}(q) = \overline{M}_i^2 + q^2\{Z_i + \mathcal{O}(q^2 - q_0^2)\}$. Here the $Z_i$ depend on the choice of $q_0^2$ through the normalization condition

$$Z_i = \frac{1}{q_0^2} \left( G_i^{-1}(q_0^2) - G_i^{-1}(0) \right)$$
$$\overline{M}_i^2 = G_i^{-1}(0) . \tag{6.11}$$

This condition, together with (6.5), also specifies the precise meaning of[8] $Z_m$. Similar definitions also apply for $Z_p$ and $Z_h$, but the momentum used can now be different from the pseudoscalar octet momentum $q_0^2$. For the definition of $Z_p$ it seems convenient to replace[9] $q_0^2$ by $q_p^2 = -m_p^2$, whereas for $Z_h$ one may use $q_h^2 = -m_h^2$. We note that our definition (6.11) of $Z_i$ also specifies the precise meaning of the couplings $\omega_m$ and $\omega_{pm}$ in (6.2). They multiply three–point functions at zero momentum for[10] $h$ and momenta given by $q_0^2$ for $p$ and $m$. All these specifications are irrelevant as long as only terms with two derivatives are included. The conceptual setting becomes crucial, however, once we go beyond this approximation and include higher derivative terms.

In fact, the definition $M_i^2 = \overline{M}_i^2/Z_i$ yields the physical pole masses $M_i^2$ only if we replace in (6.11) the common momentum $q_0^2$ by the individual locations of the poles at $q_i^2 = -M_i^2$. This involves corrections of the inverse propagators $\sim (q_i^2 - q_0^2)$. We discuss these additional contributions from higher derivative terms more explicitly in Sect. 12. Here we note only the following general properties: Except for the mixing angle the higher derivative corrections can be absorbed completely in the wave function renormalization constants $Z_i$. They lead to additional terms on the right hand side of (6.2) which are propor-

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[8] For an expression of the propagators in terms of the fields $\Phi$ or $\Phi_p$ the $Z_i$ stand for $Z_{\pi^\pm} Z_m$ etc. and $\overline{M}_i^2$ should be replaced by $\overline{M}_{\pi^\pm}^2 Z_m$ etc

[9] For convenience we will later also use $q_p^2 = -M_\eta^2$

[10] The normalization of $h$, however, is specified by $Z_h$ and therefore adapted to the behavior of the scalar propagator in the vicinity of its pole

tional to $(q_i^2 - q_0^2)$ according to

$$G_i^{-1}(q) = \overline{M}_i^2 + \overline{Z}_i q^2 + \overline{H}_i(q^2 - q_0^2)q^2$$

$$+ \tfrac{1}{2}\overline{H}_i^{(2)}(q^2 - q_0^2)^2 q^2 + \dots \qquad (6.12)$$

$$= \overline{M}_i^2 + Z_i q^2 + H_i(q^2 - q_i^2)q^2 + \dots \, .$$

Here the quantities $\overline{Z}_i$ correspond to a normalization at $q_0^2$ and are given by (6.2) whereas the true wave function renormalizations $Z_i$ are defined at $q_i^2 = -M_i^2$, with

$$Z_i = \overline{Z}_i + \Delta Z_i$$
$$\Delta Z_i = \overline{H}_i(q_i^2 - q_0^2) + \tfrac{1}{2}\overline{H}_i^{(2)}(q_i^2 - q_0^2)^2 + \dots \, . \qquad (6.13)$$

The difference $q_i^2 - q_0^2$ is given by pseudoscalar mass differences and is therefore linear in the quark masses. More precisely, it is again linear in $v$ and $w$. The Taylor expansion of $G^{-1}(q) - G^{-1}(0)$ around $q_0^2$ seems most reliable for $\sqrt{-q_0^2}$ somewhere inbetween the kaon and pion masses. We will choose the renormalization scale for $Z_m$ as

$$q_0^2 = -\frac{1}{3}\left(M_{K^\pm}^2 + M_{K^0}^2 + M_{\pi^\pm}^2\right) \qquad (6.14)$$

such that (6.5) remains valid. This implies that $\Delta Z_\pi = -(\Delta Z_{K^\pm} + \Delta Z_{K^0})$ and we can absorb the higher derivative effects partially in a redefinition of an effective

$$\overline{\omega}_m = \omega_m - \frac{\Delta Z_\pi}{v} \qquad (6.15)$$

leading to

$$\begin{aligned} Z_\pi &= 1 - \overline{\omega}_m v \\ Z_{K^\pm} &= 1 + \tfrac{1}{2}\overline{\omega}_m(v + w) \\ Z_{K^0} &= 1 + \tfrac{1}{2}\overline{\omega}_m(v - w) \\ Z_8 &= 1 + \overline{\omega}_m v + K_8 \, . \end{aligned} \qquad (6.16)$$

In terms of $\overline{\omega}_m$ only $Z_8$ receives an additional correction

$$K_8 = \Delta Z_8 - 2\Delta Z_K \, . \qquad (6.17)$$

This yields the complete expression for $M_8^2$ to quadratic order in the quark masses

$$\begin{aligned} M_8^2 = &\ \tfrac{1}{3}(1 - K_8)\left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right) \\ &- \tfrac{1}{3}\overline{\omega}_m v\left(M_{K^\pm}^2 + M_{K^0}^2 - 2M_{\pi^\pm}^2\right) \\ &- \tfrac{1}{4}\lambda_2\left[(f_{K^\pm} + f_{K^0} - 2f_\pi)\right. \\ &\left. - \tfrac{1}{4}\overline{\omega}_m v(f_{K^\pm} + f_{K^0} + 4f_\pi)\right]^2 \, . \end{aligned} \qquad (6.18)$$

For a computation of $M_\eta$ to quadratic order in the quark masses one needs, in addition, the complete expression for the mixing angle $\theta_p$ to linear order in the quark masses.

In order to get a rough estimate of the size of the higher derivative corrections one may assume that the true inverse propagator does not deviate by more than 10% from the lowest order form $M^2 + q^2$ over a momentum

range between $-M_\pi^2$ and $-M_\eta^2$. This results in a typical bound $\left|\overline{H}_m\right| \lesssim 0.1\left|q_0^2\right|^{-1}$ and we expect the higher derivative corrections to be unimportant. Furthermore, within a systematic derivative expansion we can take to leading order a common $\overline{H}_m$ for the whole octet. This yields

$$K_8 = \overline{H}_m\left(\tfrac{4}{3}M_K^2 - M_\eta^2 - \tfrac{1}{3}M_\pi^2\right) \, . \qquad (6.19)$$

Inserting the leading order for $M_\eta^2$ on the right hand side of (6.19) we find

$$K_8 \sim \mathcal{O}(\Delta^2) \, . \qquad (6.20)$$

We conclude that $K_8$ gives corrections to the pseudoscalar masses which are formally cubic in the $m_q$.

The deviation of $Z_\pi$, $Z_{K^\pm}$, etc. from one can lead to a sizeable change of the infered value for $v$ and therefore to important modifications of the values of the couplings of the linear sigma model. Inserting (6.16) into (6.9) and (5.6) yields to leading order (neglecting isospin violation)

$$\begin{aligned} \left(\tfrac{Z_m}{Z_h}\right)^{\frac{1}{2}} v &= \tfrac{2}{3}\left[f_K - f_\pi - \tfrac{1}{4}\overline{\omega}_m v(f_K + 2f_\pi)\right] \\ \sigma_0 &= \tfrac{1}{6}\left[2f_K + f_\pi - \tfrac{1}{2}\overline{\omega}_m v(f_K - f_\pi)\right] \, . \end{aligned} \qquad (6.21)$$

We observe that the correction to the relation (2.6) is of the order $(Z_h/Z_m)^{1/2}\overline{\omega}_m \sigma_0$. It is not suppressed by quark mass terms. On the other hand, the correction to $\sigma_0$ is of second order in the quark masses and therefore small. Taking $\overline{\omega}_m v = -0.20$ (see Sect. 13) the relations (6.21) result in

$$\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v \simeq 23.3 \text{ MeV} \, , \quad \sigma_0 \simeq 53.8 \text{ MeV} \qquad (6.22)$$

to be compared with (2.7) and (5.6). We conclude that the quark mass corrections to the kinetic terms are important for a quantitative understanding of the linear sigma model! These corrections also matter for a determination of the decay constant $f_{K^0}$. We first notice the change in $w$ induced by $\overline{\omega}_m v = -0.20$ in (2.12). Using (6.16) we find

$$\begin{aligned} \left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} w &= -\frac{3}{2}\frac{\left(\tfrac{Z_m}{Z_h}\right)^{\frac{1}{2}} v}{(1 - \overline{\omega}_m v)}\frac{(M_{K^0}^2 - M_{K^\pm}^2)}{(M_K^2 - M_\pi^2)} \\ &\simeq -0.67 \text{ MeV} \, . \end{aligned} \qquad (6.23)$$

This yields

$$f_{K^0} - f_{K^\pm} = -\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} w$$

$$\times \left[\left(1 + \tfrac{1}{2}\overline{\omega}_m v\right)^{\frac{1}{2}} + \frac{f_{K^\pm}}{2\left(\tfrac{Z_m}{Z_h}\right)^{\frac{1}{2}} v}\frac{\overline{\omega}_m v}{\left(1 + \tfrac{1}{2}\overline{\omega}_m v\right)}\right]$$

$$\simeq 0.28 \text{ MeV} \qquad (6.24)$$

to be compared with (2.12), $f_{K^0} - f_{K^\pm} = 0.47$ MeV, for $\overline{\omega}_m v = 0$. The quark mass corrections to the kinetic terms reduce the isospin violation of the decay constants significantly!

Let us next turn to the term $\sim \omega_{pm}\partial_\mu p \, \mathrm{Tr}\, h\partial^\mu m$ in (6.1) which influences the octet–singlet mixing angle $\theta_p$. This term leads to an off–diagonal kinetic term

$$\hat{\omega}\partial^\mu m_8 \partial_\mu p = -\sqrt{3}\omega_{pm}v\partial^\mu m_8 \partial_\mu p \ . \qquad (6.25)$$

Using $m_{R8} = Z_8^{\frac{1}{2}} m_8$ the inverse propagator for the fields $p$, $m_{R8}$ is given in momentum space by the matrix

$$G^{-1} = \qquad\qquad\qquad\qquad\qquad\qquad (6.26)$$

$$\begin{pmatrix} z_p(q^2)q^2 + M_p^2, & Z_8^{-\frac{1}{2}}\left(\hat{\omega}(q^2)q^2 + M_{8p}^2\right) \\ Z_8^{-\frac{1}{2}}\left(\hat{\omega}(q^2)q^2 + M_{8p}^2\right), & z_8(q^2)q^2 + M_8^2 \end{pmatrix} \ .$$

Here $M_{8p}^2$ is given to quadratic order in $m_q$ by (5.4), whereas $M_p^2 = m_p^2$, (5.12), and $M_8^2$ obeys (6.7). We parameterize corrections from higher derivative terms for the diagonal elements of (6.26) by the functions $z_p(q^2)$, $z_8(q^2)$. Our normalization of $Z_p$ and $Z_8$ is chosen such that $z_p(-M_\eta^2) = 1$, $z_8(-M_\eta^2) = 1$. Higher derivative corrections to the off–diagonal elements of (6.26) result in an effective $q^2$–dependence of $\hat{\omega}$. For $q^2 = -M_\eta^2$ this correction is formally of third order in the quark masses, whereas for $q^2 = -M_{\eta'}^2$ it is of first order (since $M_{\eta'}^2 - M_\eta^2$ is counted as being of zeroth order). In real life, however, $M_{\eta'}^2$ and $M_\eta^2$ are separated by a factor less than four. With the reasonable assumption that there is no dramatic $q^2$–dependence of $\hat{\omega}$ we will treat $\hat{\omega}q^2$ as a term linear in the quark masses for both $-q^2 = M_\eta^2$ and $M_{\eta'}^2$, and correspondingly count the higher derivative corrections as terms quadratic in the quark masses. For a lowest order estimate they can therefore be neglected, $z_p(q^2) = z_8(q^2) = 1$, $\hat{\omega}(q^2) = \hat{\omega}$.

The diagonalization of (6.26) can not be performed independently of $q^2$ anymore. As a consequence, the effective octet–singlet mixing angle $\theta_p$ will depend on $q^2$:

$$\tan \theta_p(q^2) =$$
$$\frac{M_8^2 - M_p^2 + \sqrt{(M_8^2 - M_p^2)^2 + 4(M_{8p}^2 + \hat{\omega}q^2)^2 Z_8^{-1}}}{2(M_{8p}^2 + \hat{\omega}q^2)Z_8^{-\frac{1}{2}}} \ .$$
$$(6.27)$$

The propagator in a diagonal basis corresponds to the eigenvalues of $G$, i.e.

$$G_{\eta',\eta}^{-1}(q^2) = q^2 + \frac{1}{2}\left(M_p^2 + M_8^2\right) \qquad\qquad (6.28)$$
$$\pm \left[\frac{1}{4}\left(M_p^2 - M_8^2\right)^2 + \frac{M_{8p}^4}{Z_8} + \frac{\hat{\omega}^2}{Z_8}q^4 + 2\frac{\hat{\omega}M_{8p}^2}{Z_8}q^2\right]^{\frac{1}{2}} \ .$$

The masses $M_{\eta'}^2$ and $M_\eta^2$ are given by the location of the poles of $G$ for negative $q^2$:

$$M_{\eta',\eta}^2 = \left(1 - \frac{\hat{\omega}^2}{Z_8}\right)^{-1}\left[\frac{1}{2}\left(M_p^2 + M_8^2\right) - \frac{\hat{\omega}}{Z_8}M_{8p}^2 \right.$$
$$\pm \left\{\frac{1}{4}\left(M_p^2 - M_8^2\right)^2 + \frac{M_{8p}^4}{Z_8}\right.$$
$$\left.\left. - \frac{\hat{\omega}}{Z_8}M_{8p}^2\left(M_p^2 + M_8^2\right) + \frac{\hat{\omega}^2}{Z_8}M_p^2 M_8^2\right\}^{\frac{1}{2}}\right] \ .$$
$$(6.29)$$

Expanding to quadratic order in the quark masses (with $\hat{\omega}$ and $M_{8p}^2$ linear in $m_q$) one finds the relations

$$M_{\eta'}^2 + M_\eta^2 = M_p^2 + M_8^2 - 2\hat{\omega}M_{8p}^2$$
$$+ \hat{\omega}^2\left(M_p^2 + M_8^2\right) \qquad\qquad (6.30)$$

$$M_{\eta'}^2 - M_\eta^2 = M_p^2 - M_8^2 + \hat{\omega}^2\left(M_p^2 - M_8^2\right) + \frac{2}{M_p^2 - M_8^2}$$
$$\times \left[M_{8p}^4 - \hat{\omega}M_{8p}^2\left(M_p^2 + M_8^2\right) + \hat{\omega}^2 M_p^2 M_8^2\right]$$
$$= M_{\eta'}^2 + M_\eta^2 - 2M_8^2 + \frac{2M_{8p}^4}{M_{\eta'}^2 + M_\eta^2 - 2M_8^2} \ .$$
$$(6.31)$$

It is remarkable that the relation between $M_{\eta'}^2 - M_\eta^2$ and $M_{\eta'}^2 + M_\eta^2$ becomes independent of $\hat{\omega}$ in this approximation.

For an experimental determination of the octet–singlet mixing through the decay of $\eta'$ or $\eta$ into two photons the relevant quantities will be $\theta_p(q^2 = -M_{\eta'}^2)$ or $\theta_p(q^2 = -M_\eta^2)$, respectively. The mixing angle depends strongly on $\hat{\omega}$ if $\hat{\omega}M_{\eta'}^2$ is of the same order of magnitude as $M_{8p}^2$. This leads to a sizeable dependence on $q^2$. If one intends to compute the octet–singlet mixing angle to quadratic order in the quark masses one needs in addition contributions to $\hat{\omega}$ quadratic in $m_q$. They arise from a modification of the kinetic term through

$$\mathcal{L}_{\mathrm{kin}}^{(2)} = \frac{1}{2}\omega'_{pm}\partial_\mu p \, \mathrm{Tr}\, h^2\partial^\mu m + \dots \ . \qquad (6.32)$$

This leads to a second order correction for $\hat{\omega}$

$$\hat{\omega}(q^2) = -\sqrt{3}\left[\omega_{pm}v + \omega'_{pm}v^2\right]\frac{f_\omega(q^2)}{f_\omega(-m_m^2)} \qquad (6.33)$$

where $f_\omega(q^2)$ contains the higher derivative corrections where $f_\omega(-M_\eta^2) = 1$ and $f_\omega(-m_m^2)$ reflects the normalization of $\omega_{pm}$, $\omega'_{pm}$ at $q^2 = -m_m^2$. Also the deviation of $Z_8$ from one has to be included to this order. We note that for $z_p(q^2) = z_8(q^2)$ the formula (6.27) remains valid if $\hat{\omega}$ is replaced by $\hat{\omega}(q^2)$, (6.33).

## 7 $M_\eta$ and $M_{\eta'}$ to quadratic order

We are now ready to address the question of mass relations for the pseudoscalar mesons to quadratic order in the quark masses. First, we replace the coupling $\nu$, or equivalently $M_p^2$, by $M_{\eta'}^2$ as a phenomenological input parameter. Our aim is to compute $M_\eta^2$ and the mixing angle $\theta_p$ as functions of $M_{\eta'}^2$, $M_{\pi^\pm}^2$, $M_{K^\pm}^2$, $M_{K^0}^2$, $f_K$ and $f_\pi$. For this purpose we use the relations (6.31) and (6.27) which involve the quantities $M_8^2$, $M_{8p}^2 Z_8^{-1/2}$ and $\hat{\omega}Z_8^{-1/2}$. The difference $M_{\eta'}^2 - M_\eta^2$ is to quadratic order independent of $\hat{\omega}$. The dependence on $Z_8$ arises only indirectly through the correction $\sim K_8$ in $M_8^2$ (6.18) and gives no correction to quadratic order in the quark masses, (6.20). The mixing angle depends already to linear order on $\hat{\omega}$, but the
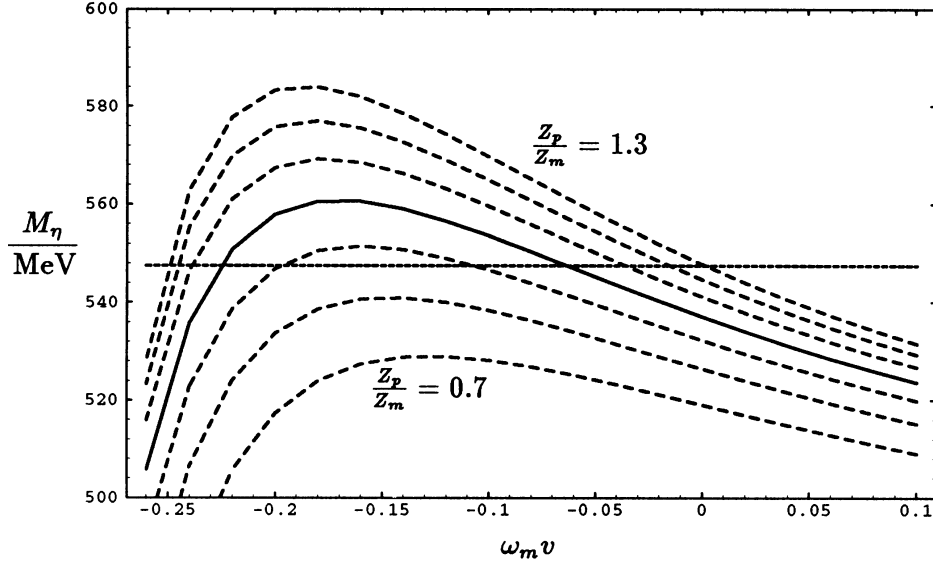
**Fig. 1.** The plot shows $M_\eta$ as a function of $\omega_m v$ for various values of $Z_p/Z_m$ and $\omega_m = \overline{\omega}_m$. The *solid line* corresponds to $Z_p/Z_m = 1$ and the difference in $Z_p/Z_m$ between two adjacent lines is 0.1. The *horizontal dotted line* indicates the experimental value $M_\eta \simeq 547.5$ MeV

difference between $Z_8$ and one is only needed to quadratic order. For the octet mass term $M_8^2$ we use (6.18) with

$$12\delta_9 v^2 = -\frac{1}{4}\lambda_2 \left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)^2 \qquad (7.1)$$

and determine $\lambda_2$ by (5.14), with $M_{\eta'}^2$ replaced by the lowest order expression $M_{\eta'}^2 + M_\eta^2 - \frac{1}{3}\left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right)$. To the order relevant for our estimate $M_{8p}^2$ is given by (5.19). Inserting this into the second expression (6.31) one finally obtains the relation

$$M_\eta^2 = \frac{1}{3}\left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right)$$
$$+ \frac{1}{3}\left(M_{K^\pm}^2 + M_{K^0}^2 - 2M_{\pi^\pm}^2\right)\frac{\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi}{\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi}$$
$$\times \left(1 - \frac{1}{3}\left(\frac{Z_h}{Z_m}\right)^{\frac{1}{2}}\overline{\omega}_m\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)\right.$$
$$\left. - 2\frac{\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi}{\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi}\right)$$
$$- \frac{2}{9}\frac{\overline{f}_\pi^2}{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)^2}$$
$$\times \frac{\left(3M_{K^\pm}^2 + 3M_{K^0}^2 - 2M_{\pi^\pm}^2\right)^2}{\left[M_{\eta'}^2 - \frac{1}{3}\left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right)\right]}$$
$$- \frac{2}{3}\left(\frac{Z_p}{Z_m} - 1\right)\frac{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)^2}{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)^2}$$
$$\times \left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right)$$
$$\times \left[1 + \frac{2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2}{3M_{\eta'}^2 - 2M_{K^\pm}^2 - 2M_{K^0}^2 + M_{\pi^\pm}^2}\right]$$

$$- \frac{2}{3}\left(\frac{Z_m}{Z_p} - 1\right)$$
$$\times \frac{\left[\left(M_{K^\pm}^2 + M_{K^0}^2\right)\left(2\overline{f}_{K^\pm} + 2\overline{f}_{K^0} - \overline{f}_\pi\right) - M_{\pi^\pm}^2\left(\overline{f}_{K^\pm} + \overline{f}_{K^0}\right)\right]^2}{\left(3M_{\eta'}^2 - 2M_{K^\pm}^2 - 2M_{K^0}^2 + M_{\pi^\pm}^2\right)\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)^2}$$
$$- \frac{1}{3}K_8\left(2M_{K^\pm}^2 + 2M_{K^0}^2 - M_{\pi^\pm}^2\right) . \qquad (7.2)$$

For $\overline{\omega}_m = 0$ and $K_8 = 0$ we recover the mass relation (5.21) of Sect. 5. Based on (7.2) one can now evaluate the corrections $\sim \overline{\omega}_m$ and $\sim K_8$ quantitatively. Neglecting higher orders in $\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)$ one has the approximate relations

$$\frac{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)}{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)} \simeq \frac{\left(f_{K^\pm} + f_{K^0} - 2f_\pi\right)}{\left(f_{K^\pm} + f_{K^0} + f_\pi\right)} - \frac{1}{2}\overline{\omega}_m v$$
$$\frac{\overline{f}_\pi}{\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi\right)} \simeq \frac{f_\pi}{\left(f_{K^\pm} + f_{K^0} + f_\pi\right)} + \frac{1}{6}\overline{\omega}_m v . \qquad (7.3)$$

In the following we also drop $K_8$ since it is formally of third order in the quark masses. If we linearize (7.2) in $\overline{\omega}_m v = \frac{1}{3}\left(\frac{Z_h}{Z_m}\right)^{1/2}\overline{\omega}_m\left(\overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi\right)$ and $\left(\frac{Z_m}{Z_p} - 1\right)$ we find

$$M_\eta = (535.7 \text{ MeV})$$
$$\times \left[1 - 0.04\left(\frac{Z_m}{Z_p} - 1\right) - 0.511\,\overline{\omega}_m v\right] . \qquad (7.4)$$

Within the range $Z_p/Z_m = 1.0 \pm 0.2$ we see that the corrections $\sim (Z_p/Z_m - 1)$ only amount to at most 7 MeV. For a rough estimate of the size of $\overline{\omega}_m v$ we can neglect them. Comparison of (7.4) with the measured $\eta$ mass yields for $Z_p = Z_m$

$$\overline{\omega}_m v \simeq -0.043 . \qquad (7.5)$$

We should note, however, that the linearization in $\overline{\omega}_m v$ is not reliable anymore for $\overline{\omega}_m v \lesssim -0.15$. This is demonstrated in Fig. 1 where we plot $M_\eta$ as a function of $\omega_m v$ for
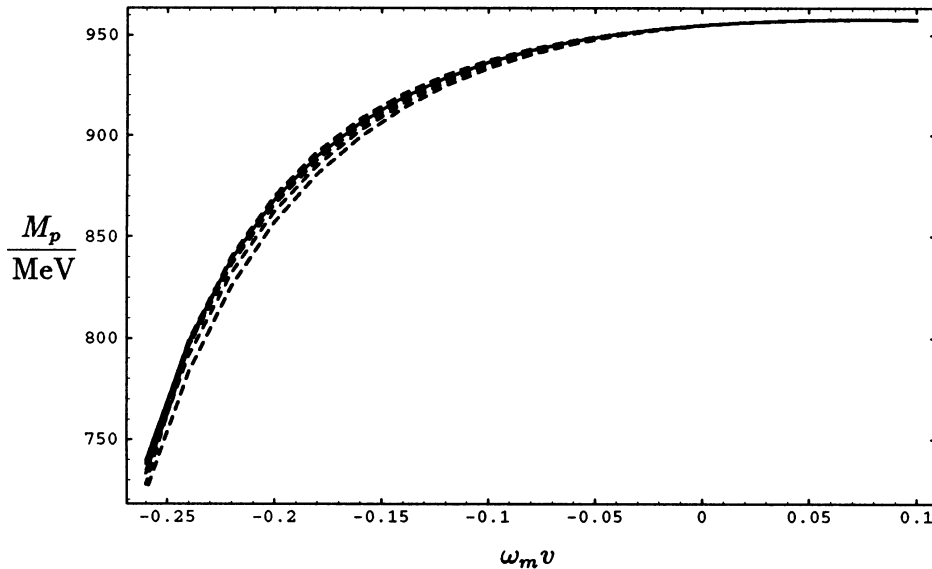
**Fig. 2.** The plot shows $M_p$ as a function of $\omega_m v$ for given $M_{\eta'} = 957.8$ MeV, $\overline{\omega}_m = \omega_m$ and $Z_p/Z_m$ varying between 0.7 and 1.3 in steps of 0.1. The solid line corresponds to $Z_p/Z_m = 1.0$

various values of $Z_p/Z_m$. For this plot we have evaluated all matrix elements in (6.26) to second order in $\Delta$ (keeping the full $\overline{\omega}_m v$–dependence, e.g., in $Z_8^{-1/2}$, though) and diagonalized the matrix without further expansion in $\Delta$. We have neglected higher derivative contributions which can not be absorbed into the wave function renormalizations, i.e. we used $z_p = z_8 = 1$, $f_\omega = 1$. For $\hat{\omega}$ we have used (8.14) and $\omega_m = \overline{\omega}_m$. The nonlinear effects due to terms which are formally $\sim \mathcal{O}(\Delta^3)$ and higher are reflected by the deviation of the curves in Fig. 1 from the tangents at $\omega_m v = 0$. Because of the important nonlinearities for $\overline{\omega}_m v \lesssim -0.15$ one finds that there is a second solution with $Z_p \simeq Z_m$, namely for $\omega_m v \simeq -0.22$. For large values of $|\omega_m v|$ the $\eta$–$\eta'$ mixing starts playing an important role. We will see below that realistic values for the decay rates $\eta \to 2\gamma$ and $\eta' \to 2\gamma$ are only consistent with this second solution for $\omega_m v$. In this region the formal quark mass expansion does not converge well anymore! This breakdown of the quark mass expansion is even more apparent in the relation between $M_{\eta'}^2$ and $M_p^2$ since linear and higher order mixing effects occur with the same sign for this quantity. We plot in Fig. 2 the value of $M_p$ as a function of $\omega_m v$ for a fixed value of $M_{\eta'} = 957.8$ MeV and various values of $Z_p/Z_m$ neglecting all higher derivative corrections to the kinetic terms. As a first observation one sees that the dependence of $M_p$ on $Z_p/Z_m$ is rather week. Furthermore, for $\omega_m v \simeq -0.22$ the ratio $M_p^2/M_{\eta'}^2$ has decreased to about 0.77 despite the fact that this effect is formally of second order in $\Delta$ if $M_p^2$ is counted as $\mathcal{O}(1)$. A partial explanation of this strong mixing effect is related to the observation that for $\omega_m v \simeq -0.22$ the value $M_p \simeq 839$ MeV is actually almost comparable to $\overline{M}_8 = 579$ MeV. (The values are for $Z_p/Z_m = 1.0$.) A counting where $\overline{M}_8^2/M_p^2 = \mathcal{O}(\Delta)$ becomes therefore doubtful. As an alternative, we may count the anomaly

contribution to $M_p^2$, i.e. $M_p^2 Z_p/Z_m - m_m^2 \simeq (742 \text{ MeV})^2$ also as an effect of order $\Delta$ since its size is not too different from the $SU(3)$ breaking induced by the mass of the strange quark. In this counting scheme all elements of the $\eta$–$\eta'$ mass matrix have the same order of magnitude. Eigenvalues to a given order in $\Delta$ involve then all matrix elements to this order and the mixing effects are naturally large. Nevertheless, since $\overline{M}_8^2/M_p^2 = 0.69$ is still substantially smaller than one the real situation is somewhere in the transition region between the two counting rules. Our approximations to order $\Delta^2$ are consistent with both ways of counting.

In the remaining sections of this paper we will gradually collect information on the quantities $Z_m/Z_p$, $\overline{\omega}_m v$, $\hat{\omega}$ and $K_8$. We should mention that the quantities $\overline{\omega}_m v$ and $K_8$ only involve properties of the effective action for the pseudoscalar octet. There is, in principle, no information which goes beyond the one contained in chiral perturbation theory. The quantities $Z_m/Z_p$ and $\hat{\omega}$ also involve the $\eta'$ and go beyond standard chiral perturbation theory. Of course, if one were able to predict directly quantities like $\overline{\omega}_m v$ one would gain in addition information on some parameters appearing in chiral perturbation theory.

## 8 Kinetic terms in the linear meson model

In this section we discuss in more detail the derivative terms in the context of the linear $\sigma$–model. Our first aim is to gain information about the size of $Z_m/Z_p$, $\omega_m$ and $\hat{\omega}$. Expanding (4.6) in powers of $\Phi_p$, $\Phi_s$ and $\chi_p$ we observe that only those terms contribute which have both derivatives acting on pseudoscalar fields. Also all fields without derivatives must be scalars. We can therefore replace in

(4.4)

$$\partial_\mu \Phi \longrightarrow \frac{i}{\sqrt{2}}\partial_\mu \Phi_p + \frac{i}{\sqrt{6}}\partial_\mu \chi_p$$
$$\partial_\mu \Phi^\dagger \longrightarrow -\frac{i}{\sqrt{2}}\partial_\mu \Phi_p - \frac{i}{\sqrt{6}}\partial_\mu \chi_p \tag{8.1}$$

and for the fields without derivatives

$$\Phi \longrightarrow \overline{\sigma}_0 + \frac{1}{\sqrt{2}}\Phi_s$$
$$\Phi^\dagger \longrightarrow \overline{\sigma}_0 + \frac{1}{\sqrt{2}}\Phi_s . \tag{8.2}$$

The terms appearing in $\mathcal{L}_{\mathrm{kin}}$ (4.6) have been selected such that the leading terms contributing to $Z_p/Z_m$, $\omega_m$ and $\omega_{pm}$ have been included. For this purpose we first classify the invariants contributing to terms with two derivatives acting on pseudoscalars which are at most linear in $\Phi_s$. The reasoning is somewhat lengthy but straightforward: By partial integration all invariants involve at most one derivative acting on a given $\Phi$. We can then distribute the two derivatives either within the same index contraction (with $\delta_{ab}$ or $\epsilon_{abc}$) with respect to $SU_L(3) \times SU_R(3)$ or among two different such structures. Traces involving six and more powers of $\Phi$ have at least one combination $\Phi^\dagger\Phi$ (or $\Phi\Phi^\dagger$) without derivatives in the chain. By a suitable definition of the invariants this may be replaced by $\left(\Phi^\dagger\Phi - \frac{1}{3}\mathrm{Tr}(\Phi^\dagger\Phi)\right)$ and such invariants contribute therefore only to higher order in the quark masses. For instance, combinations involving two factors of $\Phi^\dagger\Phi$ or $\Phi\Phi^\dagger$ (without derivatives) contribute at most to order $\Phi_s^2$. The only invariant involving a trace of six factors $\Phi^\dagger$ or $\Phi$ that may contribute linearly in $\Phi_s$ is therefore the term $\sim W_\varphi$. There are also two terms quartic in $\Phi$ with couplings $X_\varphi^+$, $X_\varphi^-$. To linear order in $\Phi_s$ the structures involving $\epsilon$–tensors are the one $\sim U_\varphi$ and terms not listed in (4.6), namely

$$\mathcal{L}_{\mathrm{kin}}(T)$$
$$= \frac{1}{2}\epsilon^{a_1 a_2 a_3}\epsilon^{b_1 b_2 b_3}\left\{ T_\varphi^{(1)}\partial^\mu\Phi_{a_1 b_1}\partial_\mu\Phi_{a_2 b_2}\right.$$
$$\times \left(\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3} - \frac{1}{3}\Phi_{a_3 b_3}\mathrm{Tr}\,\Phi^\dagger\Phi\right)$$
$$+ T_\varphi^{(2)}\partial^\mu\Phi_{a_1 b_1}\Phi_{a_2 b_2}\left(\partial_\mu\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3}\right.$$
$$\left. + \Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\partial_\mu\Phi_{a_4 b_3} - \frac{2}{3}\partial_\mu\Phi_{a_3 b_3}\mathrm{Tr}\,\Phi^\dagger\Phi\right)$$
$$+ T_\varphi^{(3)}\partial^\mu\Phi_{a_1 b_1}\Phi_{a_2 b_2}$$
$$\times \left(\Phi_{a_3 b_4}\partial_\mu\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3} + \frac{1}{3}\partial_\mu\Phi_{a_3 b_3}\mathrm{Tr}\,\Phi^\dagger\Phi\right)$$
$$+ T_\varphi^{(4)}\left(\Phi_{a_1 b_1}\Phi_{a_2 b_2}\partial^\mu\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\partial_\mu\Phi_{a_4 b_3}\right.$$
$$\left. + \frac{1}{3}\Phi_{a_1 b_1}\Phi_{a_2 b_2}\Phi_{a_3 b_3}\mathrm{Tr}\,\partial^\mu\Phi^\dagger\partial_\mu\Phi\right)$$
$$+ T_\varphi^{(5)}\Phi_{a_1 b_1}\Phi_{a_2 b_2}\left(\partial^\mu\Phi_{a_3 b_4}\partial_\mu\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3}\right.$$
$$\left. + \Phi_{a_3 b_4}\partial^\mu\Phi_{b_4 a_4}^\dagger\partial_\mu\Phi_{a_4 b_3} + 2\partial^\mu\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\partial_\mu\Phi_{a_4 b_3}\right)$$
$$+ \left(\Phi \leftrightarrow \Phi^\dagger\right)\Bigg\} \tag{8.3}$$

These invariants do not contribute to $Z_m$ or $Z_p$ but they may contribute to $\omega_m$ or $\omega_{pm}$. We treat $\mathcal{L}_{\mathrm{kin}}(T)$ as a higher order correction to the term $\sim U_\varphi$ and neglect this piece in the following. Next we turn to the case where the two derivatives act within two different $SU_L(3) \times SU_R(3)$ invariant index structures. Since the invariants $\rho$, $\tau_2$, $\tau_3$ and $\xi$ are at least quadratic in the pseudoscalar fields the terms $\sim \partial_\mu\rho\partial^\mu\rho$, $\partial_\mu\tau_2\partial^\mu\tau_2$, $\partial_\mu\tau_3\partial^\mu\tau_3$, or $\partial_\mu\xi\partial^\mu\xi$ do not contribute to $Z_m$, $Z_p$, $\omega_m$ or $\hat{\omega}$. The same holds for mixed terms like $\partial_\mu\rho\partial^\mu\xi$ and for index structures of the type $\mathrm{Tr}\,\Phi^\dagger\partial^\mu\Phi\,\mathrm{Tr}\,\Phi^\dagger\Phi\partial_\mu\Phi^\dagger\Phi$ etc. On the other hand, the pseudoscalar invariant $\omega = i(\det\Phi - \det\Phi^\dagger)$ contains a term linear in the pseudoscalar fields

$$\partial_\mu\omega = -\sqrt{6}\overline{\sigma}_0^2\partial_\mu\chi_p + \frac{1}{2\sqrt{6}}\mathrm{Tr}\,\Phi_s^2\partial_\mu\chi_p$$
$$+ \overline{\sigma}_0\mathrm{Tr}\,\Phi_s\partial_\mu\Phi_p - \frac{1}{\sqrt{2}}\mathrm{Tr}\,\Phi_s^2\partial_\mu\Phi_p + \dots . \tag{8.4}$$

More generally, we can construct invariants with a possible contribution to $Z_p/Z_m$, $\omega_m$ or $\omega_{pm}$ by Lorentz contraction of $\mathcal{CP}$–odd factors like $\partial_\mu\omega$,

$$i\epsilon^{a_1 a_2 a_3}\epsilon^{b_1 b_2 b_3}\left\{\partial_\mu\Phi_{a_1 b_1}\Phi_{a_2 b_2}\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3}\right.$$
$$\left. - \partial_\mu\Phi_{a_1 b_1}^\dagger\Phi_{a_2 b_2}^\dagger\Phi_{a_3 b_4}^\dagger\Phi_{b_4 a_4}\Phi_{a_4 b_3}^\dagger\right\},$$
$$i\epsilon^{a_1 a_2 a_3}\epsilon^{b_1 b_2 b_3}\partial_\mu\left\{\Phi_{a_1 b_1}\Phi_{a_2 b_2}\Phi_{a_3 b_4}\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3}\right.$$
$$\left. - \Phi_{a_1 b_1}^\dagger\Phi_{a_2 b_2}^\dagger\Phi_{a_3 b_4}^\dagger\Phi_{b_4 a_4}\Phi_{a_4 b_3}^\dagger\right\},$$
$$i\epsilon^{a_1 a_2 a_3}\epsilon^{b_1 b_2 b_3}\left\{\Phi_{a_1 b_1}\Phi_{a_2 b_2}\Phi_{a_3 b_4}\partial_\mu\Phi_{b_4 a_4}^\dagger\Phi_{a_4 b_3}\right.$$
$$\left. - \Phi_{a_1 b_1}^\dagger\Phi_{a_2 b_2}^\dagger\Phi_{a_3 b_4}^\dagger\partial_\mu\Phi_{b_4 a_4}\Phi_{a_4 b_3}^\dagger\right\}.$$

Within these structures we may replace subsets $\Phi\Phi^\dagger\Phi$ by the combination $\Phi\Phi^\dagger\Phi - \frac{1}{3}\Phi\,\mathrm{Tr}\,\Phi^\dagger\Phi$ which is $\sim \Phi_s$. We will keep here only the term which is $\sim \partial_\mu\omega\partial^\mu\omega$ and consider the other contractions as higher order corrections to the term $\sim \tilde{V}_\varphi$. The phenomenological analysis below will indicate that the "anomaly" terms $\sim U_\varphi$ and $\sim \tilde{V}_\varphi$ are not very important. This justifies to neglect corrections to them. We have also included in (4.6) some invariants which do not contribute to $Z_p/Z_m$, $\omega_m$ and $\omega_{pm}$. This is partly for the purpose to demonstrate that not much can be learned about the ratio $Z_h/Z_m$ from exploiting the symmetries of the linear sigma model.

The contribution of the term $\sim \tilde{V}_\varphi$ to $Z_m$, $Z_p$ and $\hat{\omega}$ can be read off from

$$\mathcal{L}_{\mathrm{kin}}(\tilde{V}_\varphi) = \frac{1}{2}\tilde{V}_\varphi\partial^\mu\omega\partial_\mu\omega$$
$$= 3\tilde{V}_\varphi\overline{\sigma}_0^4\partial^\mu\chi_p\partial_\mu\chi_p - \sqrt{6}\tilde{V}_\varphi\overline{\sigma}_0^3\partial^\mu\chi_p\,\mathrm{Tr}\,\Phi_s\partial_\mu\Phi_p$$
$$+ \sqrt{3}\tilde{V}_\varphi\overline{\sigma}_0^2\partial^\mu\chi_p\,\mathrm{Tr}\,\Phi_s^2\partial_\mu\Phi_p + \dots . \tag{8.5}$$

Next we turn to the terms $\sim X_\varphi^-$ and $X_\varphi^+$ in (4.6). Whereas $X_\varphi^+$ gives no contribution linear in the quark masses the term $\sim X_\varphi^-$ yields

$$\mathcal{L}_{\mathrm{kin}}(X_\varphi^-) = \frac{1}{2}X_\varphi^-\overline{\sigma}_0^2\left(\mathrm{Tr}\,\partial^\mu\Phi_p\partial_\mu\Phi_p + \partial^\mu\chi_p\partial_\mu\chi_p\right)$$

$$+\frac{1}{\sqrt{2}}X_\varphi^-\overline\sigma_0\operatorname{Tr}\Phi_s\partial^\mu\Phi_p\partial_\mu\Phi_p$$

$$+\frac{2}{\sqrt{6}}X_\varphi^-\overline\sigma_0\partial^\mu\chi_p\operatorname{Tr}\Phi_s\partial_\mu\Phi_p$$

$$+\frac{1}{2\sqrt{3}}X_\varphi^-\partial^\mu\chi_p\operatorname{Tr}\Phi_s^2\partial_\mu\Phi_p\,. \tag{8.6}$$

Similarly, one obtains

$$\mathcal{L}_{\text{kin}}(W_\varphi)=\frac{1}{2\sqrt{2}}W_\varphi\overline\sigma_0^3\operatorname{Tr}\Phi_s\partial^\mu\Phi_p\partial_\mu\Phi_p$$

$$+\frac{1}{\sqrt{6}}W_\varphi\overline\sigma_0^3\partial^\mu\chi_p\operatorname{Tr}\Phi_s\partial_\mu\Phi_p$$

$$+W_\varphi\overline\sigma_0^2\left\{\frac{3}{8}\operatorname{Tr}\Phi_s^2\partial^\mu\Phi_p\partial_\mu\Phi_p\right.$$

$$-\frac{1}{24}\operatorname{Tr}\Phi_s^2\operatorname{Tr}\partial^\mu\Phi_p\partial_\mu\Phi_p$$

$$+\frac{1}{4}\operatorname{Tr}\Phi_s\partial^\mu\Phi_p\Phi_s\partial_\mu\Phi_p+\frac{5}{4\sqrt{3}}\partial^\mu\chi_p\operatorname{Tr}\Phi_s^2\partial_\mu\Phi_p$$

$$\left.+\frac{1}{6}\operatorname{Tr}\Phi_s^2\partial^\mu\chi_p\partial_\mu\chi_p\right\}+\dots\,. \tag{8.7}$$

For the piece $\sim U_\varphi$ we use

$$\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}=\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}+\delta_{a_1b_2}\delta_{a_2b_3}\delta_{a_3b_1}$$

$$+\delta_{a_1b_3}\delta_{a_2b_1}\delta_{a_3b_2}-\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_2}$$

$$-\delta_{a_1b_3}\delta_{a_2b_2}\delta_{a_3b_1}-\delta_{a_1b_2}\delta_{a_2b_1}\delta_{a_3b_3} \tag{8.8}$$

and find

$$\mathcal{L}_{\text{kin}}(U_\varphi)=U_\varphi\overline\sigma_0\left(\frac{1}{2}\operatorname{Tr}\partial^\mu\Phi_p\partial_\mu\Phi_p-\partial^\mu\chi_p\partial_\mu\chi_p\right)$$

$$-\frac{1}{\sqrt{2}}U_\varphi\operatorname{Tr}\Phi_s\partial^\mu\Phi_p\partial_\mu\Phi_p$$

$$+\frac{1}{\sqrt{6}}U_\varphi\partial^\mu\chi_p\operatorname{Tr}\Phi_s\partial_\mu\Phi_p\,. \tag{8.9}$$

In our truncation no other second derivative invariant (except the one $\sim Z_\varphi$) contributes to $Z_m$, $Z_p$ $\omega_m$, $\omega_{pm}$ or $\omega'_{pm}$. Combining (8.5), (8.6) and (8.9) with the term $\sim Z_\varphi$ and comparing with (4.9), (6.1), (6.32) and (6.33) yields

$$Z_m=Z_\varphi+X_\varphi^-\overline\sigma_0^2+U_\varphi\overline\sigma_0$$

$$Z_p=Z_\varphi+X_\varphi^-\overline\sigma_0^2+6\tilde V_\varphi\overline\sigma_0^4-2U_\varphi\overline\sigma_0$$

$$\omega_m=\left(X_\varphi^-\overline\sigma_0+\frac{1}{2}W_\varphi\overline\sigma_0^3-U_\varphi\right)Z_h^{-\frac{1}{2}}Z_m^{-1}$$

$$\omega_{pm}=\left(\frac{2}{\sqrt{6}}X_\varphi^-\overline\sigma_0+\frac{1}{\sqrt{6}}W_\varphi\overline\sigma_0^3-\sqrt{6}\tilde V_\varphi\overline\sigma_0^3+\frac{1}{\sqrt{6}}U_\varphi\right)$$

$$\times Z_p^{-\frac{1}{2}}Z_h^{-\frac{1}{2}}Z_m^{-\frac{1}{2}}$$

$$\omega'_{pm}=\left(\frac{1}{2\sqrt{6}}X_\varphi^-+\frac{5}{4\sqrt{6}}W_\varphi\overline\sigma_0^2+\frac{\sqrt{6}}{2}\tilde V_\varphi\overline\sigma_0^2\right)$$

$$\times Z_p^{-\frac{1}{2}}Z_h^{-1}Z_m^{-\frac{1}{2}}$$

$$\hat\omega=-\left(\overline f_{K^\pm}+\overline f_{K^0}-2\overline f_\pi\right)\left[\frac{1}{3\sqrt{2}}U_\varphi+\frac{2}{3\sqrt{2}}X_\varphi^-\overline\sigma_0\right.$$

$$+\frac{1}{3\sqrt{2}}W_\varphi\overline\sigma_0^3-\sqrt{2}\tilde V_\varphi\overline\sigma_0^3+\frac{\overline f_{K^\pm}+\overline f_{K^0}-2\overline f_\pi}{\overline f_{K^\pm}+\overline f_{K^0}+\overline f_\pi}$$

$$\times\left(\frac{1}{3\sqrt{2}}X_\varphi^-\overline\sigma_0+\frac{5}{6\sqrt{2}}W_\varphi\overline\sigma_0^3+\sqrt{2}\tilde V_\varphi\overline\sigma_0^3\right)\bigg]$$

$$\times Z_p^{-\frac{1}{2}}Z_m^{-1}\,. \tag{8.10}$$

We observe the relation

$$\omega_{pm}=\frac{1}{\sqrt{6}}\left[2\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}\omega_m\right.$$

$$\left.+\frac{1}{\sigma_0}\left(\frac{Z_m}{Z_p}-1\right)\left(\frac{Z_p}{Z_h}\right)^{\frac{1}{2}}\right] \tag{8.11}$$

which is independent of the values of $X_\varphi^-$, $W_\varphi$, $U_\varphi$ and $\tilde V_\varphi$. It leads to an estimate of $\hat\omega$ to lowest order in the quark masses:

$$\hat\omega^{(1)}=\sqrt{2}\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}$$

$$\times\left[\frac{\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}}v}{2\sigma_0}\left(\frac{Z_p}{Z_m}-1\right)-\omega_m v\right]\,. \tag{8.12}$$

If we furthermore assume $|6\tilde V_\varphi\overline\sigma_0^2|$, $|\frac{5}{2}W_\varphi\overline\sigma_0^2|\ll X_\varphi^-$ (see Sect. 10) we find[11] (for $\tilde V_\varphi=0$, $W_\varphi=0$)

$$\omega'_{pm}=\frac{1}{2\sqrt{6}\sigma_0}\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}\frac{Z_m}{Z_h}$$

$$\times\left[\omega_m\left(\frac{Z_h}{Z_m}\right)^{\frac{1}{2}}+\frac{1}{3\sigma_0}\left(1-\frac{Z_p}{Z_m}\right)\right]\,. \tag{8.13}$$

This leads to an estimate to second order in $\Delta$

$$\hat\omega^{(2)}=\hat\omega^{(1)}-\frac{1}{2\sqrt{2}}\left(\frac{Z_m}{Z_p}\right)^{\frac{1}{2}}\frac{\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}}v}{\sigma_0}$$

$$\times\left[\omega_m v+\frac{\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}}v}{3\sigma_0}\left(1-\frac{Z_p}{Z_m}\right)\right]\,. \tag{8.14}$$

These results can be used for a computation of the $\eta$–$\eta'$ mixing angle to second order in the quark masses. If we neglect the higher derivative corrections ($\overline\omega_m=\omega_m$, $K_8=0$, $z_p=z_8=f_\omega=1$) the mixing angles $\theta_p(\eta)$ and $\theta_p(\eta')$ relevant for the two photon decay of the $\eta$ and $\eta'$, respectively, depend on two parameters $\omega_m v$ and $Z_p/Z_m-1$. We plot

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[11] The coupling $\omega'_{pm}$ cannot be expressed in terms of $\omega_m$ and $Z_m-Z_p$. This reflects that the kinetic invariants are linearly independent
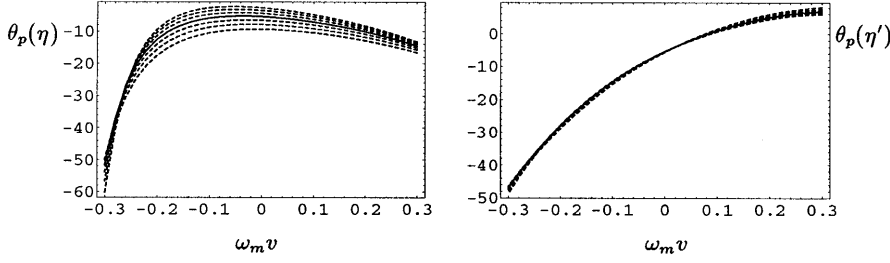
**Fig. 3.** The plots show the pseudoscalar mixing angles $\theta_p(\eta)$ and $\theta_p(\eta')$ as functions of $\omega_m v$ for various values of $Z_p/Z_m$ and $\omega_m = \overline{\omega}_m$. The *solid line* corresponds to $Z_p/Z_m = 1$ and the difference in $Z_p/Z_m$ between two adjacent lines is 0.1

in Fig. 3 these quantities as functions of $\omega_m v$ for various values of $Z_p/Z_m - 1$. Assuming $\left|\frac{Z_p}{Z_m} - 1\right| < \frac{1}{4}$ and comparing this value with $\left|\frac{1}{3}\left(\frac{Z_h}{Z_m}\right)^{1/2}\omega_m(\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi)\right| \simeq \left|\overline{\omega}_m v \frac{2f_K + f_\pi}{2(f_K - f_\pi)}\right| > \frac{1}{3}$, (7.5), we find that the first contribution is smaller than the second one. In the approximation $Z_p = Z_m$ the quantity $\hat{\omega}$ is simply related to $\omega_m v$ by

$$\hat{\omega}^{(1)} = -\sqrt{2}\omega_m v,$$

$$\hat{\omega}^{(2)} = -\sqrt{2}\omega_m v \left[1 + \frac{\left(\frac{Z_m}{Z_h}\right)^{\frac{1}{2}} v}{4\sigma_0}\right]. \qquad (8.15)$$

The existence of a relation between $\hat{\omega}$ and $\omega_m$ is crucial for the predictive power of our model since it is necessary in order to relate the $\eta$–$\eta'$ mixing angle to other observables. Within our truncation and to lowest order in $\Delta$ the relation (8.12) is a pure symmetry relation without any assumption on the values of the couplings $X_\varphi^-$, $W_\varphi$, $U_\varphi$, $\tilde{V}_\varphi$! In contrast, the kinetic term of the scalar mesons is independent of $\tilde{V}_\varphi$, $U_\varphi$ or $X_\varphi^-$, whereas $Z_h$ receives contributions $\sim X_\varphi^+$, etc. No new relations can be obtained in this way.

We observe that the couplings $U_\varphi$ and $\tilde{V}_\varphi$ violate the axial $U_A(1)$ symmetry and are therefore connected to effects from the axial anomaly. Anomaly contributions to kinetic terms of the pseudoscalar octet are often counted as higher order corrections. If we decide to do so we obtain the leading order relations

$$\begin{aligned} Z_p &= Z_m \\ \omega_m &= X_\varphi^- \overline{\sigma}_0 Z_h^{-\frac{1}{2}} Z_m^{-1}. \end{aligned} \qquad (8.16)$$

In this approximation only the kinetic term $\sim X_\varphi^-$ is essential. The importance of $X_\varphi^-$ is also manifest in the leading mixing approximation. Only this term is generated by the mixing with the divergence of the axialvector field $\partial_\mu \rho_A^\mu$ ("partial Higgs effect"). In fact, the effective action for the (pseudo)scalars contained in $\Phi$ receives contributions from the exchange of other particles. Prominent candidates are, of course, the vector and axialvector fields. We have computed in Appendix B the contributions from the exchange of the vector and axialvector meson octets as well as the associated $0^{-+}$ and $0^{+-}$ states corresponding to the divergence of the (axial)vector fields. Up to order $\Phi^4$ this

exchange only contributes to derivative terms. In terms of the couplings which contribute to the kinetic terms to linear order in $\Delta$ only $X_\varphi^-$ gets a negative contribution from the exchange of the $0^{-+}$ state whereas all other couplings remain unaffected. We find a large effect

$$\omega_m^{(\rho)} v \simeq -0.15. \qquad (8.17)$$

Comparison of this value with Figs. 1–2 indicates a large $\eta$–$\eta'$ mixing in a range where nonlinear effects in $\omega_m v$ are already important!

## 9 Decay constants of $\eta$ and $\eta'$

The decay constants $f_{\pi^0}$, $f_\eta$ and $f_{\eta'}$ are experimentally determined from the partial decay width of the $\pi^0$, $\eta$ and $\eta'$ into two photons

$$\Gamma(\eta \to 2\gamma) = \frac{\alpha^2}{64\pi^3} \frac{M_\eta^3}{f_\eta^2} \qquad (9.1)$$

and similarly for $\eta'$ and $\pi^0$. The experimental values for the decay widths are [19]

$$\begin{aligned} \Gamma(\pi^0 \to 2\gamma) &= (7.78 \pm 0.56) \text{ eV} \\ \Gamma(\eta \to 2\gamma) &= (0.46 \pm 0.04) \text{ keV} \\ \Gamma(\eta' \to 2\gamma) &= (4.26 \pm 0.19) \text{ keV} \end{aligned} \qquad (9.2)$$

which yields ($f_\pi = (92.4 \pm 0.3)$ MeV)

$$\begin{aligned} \left(\frac{f_{\pi^0}}{f_\pi}\right)^{\exp} &\simeq 1.00 \pm 0.04 \\ \left(\frac{f_\eta}{f_\pi}\right)^{\exp} &\simeq 1.06 \pm 0.05 \\ \left(\frac{f_{\eta'}}{f_\pi}\right)^{\exp} &\simeq 0.81 \pm 0.02. \end{aligned} \qquad (9.3)$$

To lowest order in the quark mass expansion the decay constant[12] $f_\eta$ is related to $f_K = f_\pi$ by $SU(3)$ symmetry (cf. Appendix A)

$$f_\eta^{(0)} = \sqrt{3} f_{\eta 8} = \sqrt{3} f_\pi. \qquad (9.4)$$

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[12] We use for $f_\eta$ and $f_{\eta'}$ the conventions of [19], with the warning that factors $\sqrt{3}$ and $\sqrt{3/8}$ appear here as compared to a perhaps more natural convention based on $SU(3)$ symmetry. In the limit of exact $SU(3)$ symmetry the quantity corresponding to $f_\pi$ is $f_{\eta 8}$

Similarly, for $Z_p = Z_m$ one finds for $f_{\eta'}$

$$f_{\eta'}^{(0)} = \sqrt{\frac{3}{8}} f_{\eta'0} = \sqrt{\frac{3}{8}} f_\pi \; . \tag{9.5}$$

The experimental values (9.3) differ substantially from this estimate. In Appendix A we have computed ((A.24)–(A.26)) the corrections to $f_{\eta 8}$ and $f_{\eta'0}$ as well as the corresponding constants for the singlet current in the $\eta$ ($f_{\eta 0}$) and the octet current in the $\eta'$ ($f_{\eta'8}$). Expanding the $Z^{1/2}$ factors to linear order in $\overline{\omega}_m v$ and neglecting additional higher derivative corrections one obtains the ratios

$$\frac{f_{\eta 8}}{f_\pi} = \frac{1}{3 f_\pi} \left[ 2 f_{K^\pm} + 2 f_{K^0} - f_\pi \right.$$
$$+ \frac{1}{2} \overline{\omega}_m v \left( f_{K^\pm} + f_{K^0} - 2 f_\pi \right)$$
$$\left. + \frac{1}{2} K_8 \left( 2 f_{K^\pm} + 2 f_{K^0} - f_\pi \right) \right]$$

$$\frac{f_{\eta'0}}{f_\pi} = \frac{1}{3 f_\pi} \left[ f_{K^\pm} + f_{K^0} + f_\pi \right.$$
$$\left. - \frac{1}{4} \overline{\omega}_m v \left( f_{K^\pm} + f_{K^0} - 2 f_\pi \right) \right] \left( \frac{Z_p}{Z_m} \right)^{\frac{1}{2}} . \tag{9.6}$$

Extracting the terms linear in the quark masses and neglecting isospin violation yields

$$\frac{f_{\eta 8}}{f_\pi} = \frac{4 f_K - f_\pi}{3 f_\pi} \simeq 1.3 \tag{9.7}$$

$$\frac{f_{\eta'0}}{f_\pi} = \frac{2 f_K + f_\pi}{3 f_\pi} \left( \frac{Z_p}{Z_m} \right)^{\frac{1}{2}} \simeq 1.15 \left( \frac{Z_p}{Z_m} \right)^{\frac{1}{2}} . \tag{9.8}$$

We note that this estimate for $f_{\eta 8}/f_\pi$ agrees well with the one obtained in chiral perturbation theory $(f_{\eta 8}/f_\pi)_{\chi PT} \simeq 1.25$ [2,20]. Denoting by $\theta_p(\eta) \equiv \theta_p(q^2 = -M_\eta^2)$ and $\theta_p(\eta') \equiv \theta_p(q^2 = -M_{\eta'}^2)$ (see (6.27)) the octet–singlet mixing angles relevant for the two photon decay of the $\eta$ and $\eta'$, respectively, one can compute [20] the effective decay constants for these decays from

$$\frac{1}{f_\eta} = \frac{1}{\sqrt{3}} \left( \frac{\cos \theta_p(\eta)}{f_{\eta 8}} - \frac{\sqrt{8} \sin \theta_p(\eta)}{f_{\eta 0}} \right) \tag{9.9}$$

$$\frac{1}{f_{\eta'}} = \frac{1}{\sqrt{3}} \left( \frac{\sin \theta_p(\eta')}{f_{\eta'8}} + \frac{\sqrt{8} \cos \theta_p(\eta')}{f_{\eta'0}} \right) \tag{9.10}$$

Here we have neglected the mixing of $\pi_0$ with $\eta$ and $\eta'$ which induces corrections to the decay constants $\sim w^2$. Since also $Z_{\pi^0}$ equals $Z_\pi$ up to corrections $\sim w^2$ we will not distinguish between $f_{\pi^0}$ and $f_\pi$ in this paper.

We plot in Fig. 4 the decay constants $f_\eta$ and $f_{\eta'}$ as functions of $\omega_m v$ for various values of $Z_p/Z_m$, assuming $\overline{\omega}_m = \omega_m$. It is obvious from these plots that no satisfactory solution exists for the value $\omega_m v \simeq -0.05$, corresponding to the first solution for $M_\eta$ (cf. Fig. 1) which

remains within the range of convergence of the quark mass expansion. On the other hand, it is encouraging that for the second solution $\omega_m v \simeq -0.22$ ($Z_p/Z_m = 1$) $f_\eta$ is already very close to the experimentally allowed window! The deviation of $f_{\eta'}$ from its experimental value by around 25% is not completely unexpected for the approximations employed so far. In fact, already the uncertainty of a first order computation of $f_\eta$ and $f_{\eta'}$ in powers of quark masses should be of the order of 10%. On top of this, the neglect of higher derivative terms is less accurate for $q^2 = -M_{\eta'}^2$ and one expects a less convergent expansion for $f_{\eta'}$. If $Z_p/Z_m$ is treated as a second free parameter we see that $\omega_m v = -0.20$, $Z_p/Z_m = 0.9$ provides a solution for which $f_\eta$ and $f_{\eta'}$ are within 10% of the experimentally allowed windows. Furthermore, these values of $\omega_m v$ are quite close to the ones estimated in Appendix B from the exchange of higher $0^{-+}$ states (cf. (8.17))! We conclude that all observations fit together in a picture with large mixing in the $\eta$–$\eta'$ sector. As discussed at the end of Sect. 7 the anomaly term comes out relatively small in this case. The naive quark mass expansion is then expected to converge well only for the flavored mesons whereas its convergence is unsatisfactory in the $\eta$–$\eta'$ sector. There it can be replaced by a modified expansion where $M_p$ is also counted as $\mathcal{O}(\Delta)$.

Apparently, the kinetic term $\sim X_\varphi^-$ which is induced by the mixing with the higher $0^{-+}$ multiplet plays a very important role in our picture. One is tempted to assume that this term dominates the rich structure in the kinetic terms for the pseudoscalars. The hypothesis that all deviations from a standard kinetic term for the pseudoscalars (i.e., (2.1) with $Z_p = Z_m$) are due to mixing leads to a highly predictive scheme. In the limit where $X_\varphi^-$ is independent of momentum this leads to $Z_p \simeq Z_m$, $\overline{\omega}_m \simeq \omega_m$. It is far from trivial that for $Z_p \simeq Z_m$ and $\overline{\omega}_m \simeq \omega_m$ there exists a value $\omega_m v = -0.20$ for which the quantities $M_\eta$, $f_\eta$ and $f_{\eta'}$ are all compatible with observation! The "robustness" of these "predictions" can be estimated from table 6 in Sect. 13 where we also give numbers for various values of $Z_p/Z_m$ and $\overline{\omega}_m/\omega_m$. A more detailed study of the effect of mixing with the higher $0^{-+}$ states can be found in Appendix C. In particular, the coupling $X_\varphi^-$ becomes effectively momentum dependent due to propagator effects. This leads to

(i) a contribution to a $q^4$–kinetic term and therefore to $(\overline{\omega}_m - \omega_m)/\omega_m \simeq 0.1$ (cf. (12.3));

(ii) an effective momentum dependence of $\hat{\omega}$ which gets multiplied by the factor $f_\omega = (M_P^2 - M_\eta^2)/(M_P^2 + q^2)$ with $M_P \gtrsim 2000$ MeV;

(iii) similar contributions to $z_p$ and $z_8$, (C.13) in (6.26);

(iv) a correction to $f_{\eta'}$ (A.26) given by (C.14);

(v) an effective $Z_p$ (normalized here for $q_0^2 = -M_\eta^2$) obeying $Z_p/Z_m = 0.99$.

Due to the large uncertainty in the value of $M_P$ relevant in the $\eta$–$\eta'$ sector we have not included these higher derivative effects in the Figs. 1–4. A quantitative discussion can be found in Sects. 12 (Fig. 8), 13 and Appendix C.
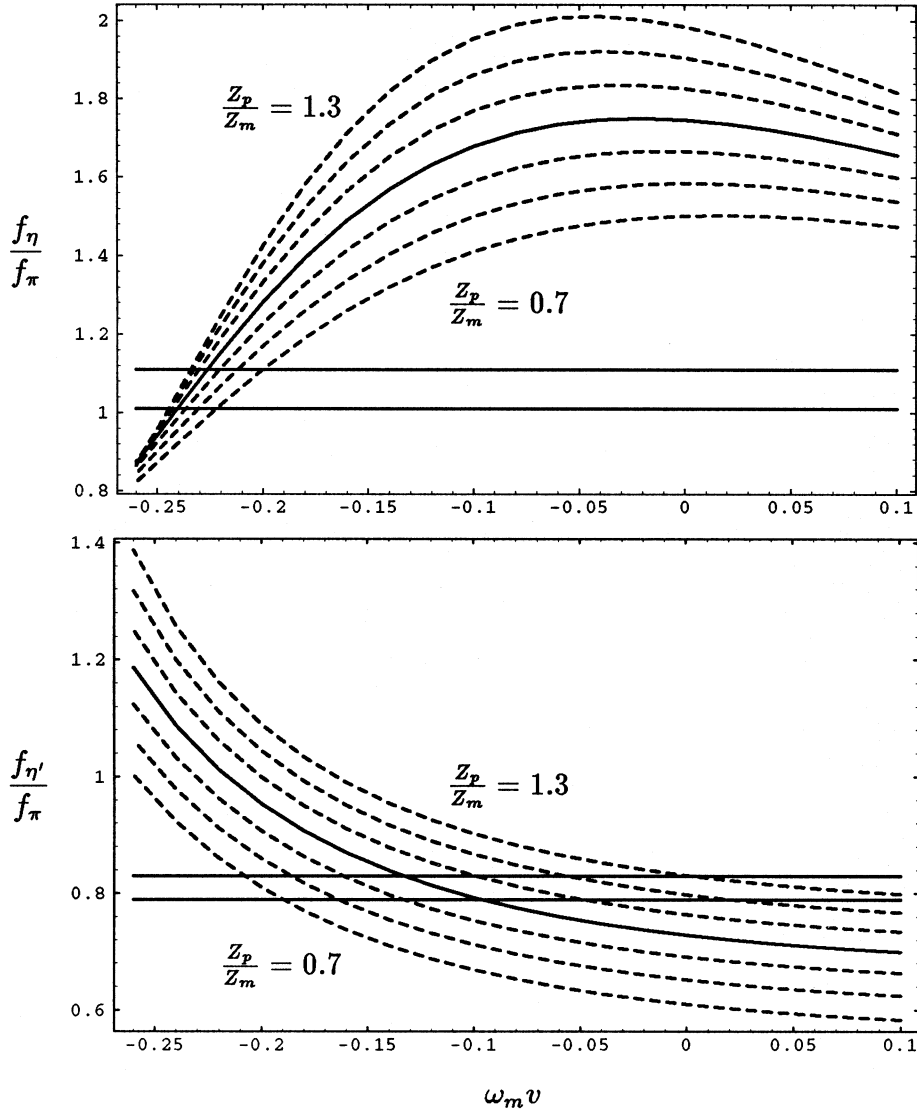
**Fig. 4.** The plots show the ratios $f_\eta/f_\pi$ and $f_{\eta'}/f_\pi$ as functions of $\omega_m v$ for various values of $Z_p/Z_m$ and $\omega_m = \overline{\omega}_m$. The *solid lines* correspond to $Z_p/Z_m = 1$ and the difference in $Z_p/Z_m$ between two adjacent lines is 0.1. The experimentally allowed windows ($1\sigma$) for both quantities are bounded by the *horizontal solid lines*

## 10 Expansion in the chiral condensate

There are various scales characteristic for the amount of spontaneous chiral symmetry breaking: the chiral condensate $|\langle \overline{\psi}\psi \rangle|^{1/3} \simeq 200$ MeV, the pion decay constant $f_\pi \simeq 90$ MeV and the constituent quark mass $m_q \simeq 300$ MeV. All these scales are typically smaller than the characteristic scale for the formation of the mesonic bound states, $k_\varphi \gtrsim 600$ MeV [5,6] or typical mesonic mass scales unrelated to the Goldstone phenomenon — the latter being around 1 GeV with the lowest one given by[13] $M_{\rho^0} = 770$ MeV. The question emerges if the typical scale appearing in the parameters $\overline{\nu}$, $X_\varphi^-$, $\overline{\lambda}_3$, etc. is larger than characteristic scales associated with chiral symmetry breaking. Could it be that chiral symmetry breaking is related to

an additional small parameter leading to a suppression of contributions with high powers of $\sigma_0$? The existence of a small parameter associated to $\sigma_0$ would enhance the predictive power of the linear sigma model since the terms with lower powers of $\sigma_0$ would dominate. Together with a derivative expansion it would allow to classify invariants according to their dimension with a suppression of higher dimensional operators.

Within the linear sigma model we observe distinct mass scales of very different origin: Whereas the expectation value $\sigma_0$ measures the strength of spontaneous chiral symmetry breaking, the scale $\nu$ indicates the size of the explicit breaking of the axial $U_A(1)$ because of the chiral anomaly. Furthermore, there are hadronic mass scales which are not directly related to chiral symmetry breaking or the axial anomaly as, for example, the string tension or the glueball masses. One expects that the last type of

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[13] The $\rho$–meson is perhaps somewhat special because of approximate gauge symmetry, see Appendix B

scales dominates the dimensionfull parameters in the effective potential and the kinetic terms in the limit of vanishing anomaly and $\sigma_0 = 0$. Since for mass terms etc. $\sigma_0$ is multiplied by some dimensionless coupling constant, a typical parameter for testing the convergence of a $\sigma_0$–expansion could be $x_\sigma = \lambda_2\sigma_0/\nu$. Using the values of the parameters determined in Sect. 5 yields $x_\sigma \simeq 0.3$ whereas including the quark mass corrections to the kinetic terms gives $x_\sigma \simeq 0.2$. This seems indeed to allow for the possibility that an expansion in powers of $\sigma_0$ does not converge too badly. We will see that this picture is confirmed by the size of other dimensionless ratios involving powers of $\sigma_0$.

In order to make a guess for the size of contributions with higher powers of $\sigma_0$ we first note that the smallness of $\sigma_0$ is partly due to the smallness of $Z_m$ [6]. Since the physics cannot depend on the choice of the scaling for the field $\Phi$, only ratios which are independent of $Z_m$ can appear in measurable quantities. This includes combinations like $\overline{\lambda}_2\overline{\sigma}_0/\overline{\nu} = \lambda_2\sigma_0/\nu$, $\overline{\lambda}_3\overline{\sigma}_0^3/\overline{\nu} = \lambda_3\sigma_0^3/\nu$, $X_\varphi^-\overline{\sigma}_0^2/Z_m$, $U_\varphi\overline{\sigma}_0/Z_m$, $\tilde{V}_\varphi\overline{\sigma}_0^4/Z_m$. The physical scales hidden in the renormalized parameters can be better appreciated if we choose (somewhat arbitrarily) a fixed $Z_m^{(0)}$ such that the dimensionless couplings are of order one, say $Z_m^{(0)} = 0.15$ such that $\overline{\lambda}_2^{(0)} \simeq \lambda_2/50$. With this scaling of the field one has $\overline{\sigma}_0^{(0)} = \sigma_0(Z_m^{(0)})^{-1/2} \simeq 137$ MeV and $\overline{\nu}^{(0)} = \nu(Z_m^{(0)})^{3/2} \simeq 540$ MeV. We will now assume that dimensionfull parameters like $\overline{\lambda}_3^{(0)}$ are given by powers of a characteristic scale which we take to be around 700 MeV, i.e. $\overline{\lambda}_3^{(0)} \simeq (700\text{ MeV})^{-2}$, $X_\varphi^{-(0)} \simeq (700\text{ MeV})^{-2}$. This suggests typical ratios $|\lambda_3\sigma_0^3/\nu| \simeq 0.015$, $|\lambda_3\sigma_0^2/\lambda_2| \simeq 0.08$, $|X_\varphi^-\overline{\sigma}_0^2/Z_m| \simeq 0.25$, $|U_\varphi\overline{\sigma}_0/Z_m| \simeq 1.3$, $|\tilde{V}_\varphi\overline{\sigma}_0^4/Z_m| \simeq 0.01$. Obviously, these numbers can only be used as rough guesses. There may be additional small coefficients — this is obviously necessary for the contribution $\sim U_\varphi$ if $Z_p$ is in the vicinity of $Z_m$ (see (8.10) — or relatively large group theoretical or dynamical factors. If the contribution from $U_\varphi$ does not dominate $\omega_m$ we may estimate $|X_\varphi^-\overline{\sigma}_0^2/Z_m| \simeq |\omega_m Z_h^{1/2}\overline{\sigma}_0| \simeq 0.5$ (for $\omega_m v = -0.20$) which is somewhat larger but still compatible with the above guess. We conclude that the $\sigma_0$–expansion converges at best slowly. For low powers of $\sigma_0$ group theoretical factors or dynamically small quantities (i.e. $Z_m^{(0)}$, $U_\varphi\overline{\sigma}_0$) remain very relevant. Nevertheless, we find it very unlikely that terms with high powers of $\sigma_0$ dominate those with low powers. Even the conservative assumption that terms with high powers of $\sigma_0$ are bounded in size by the strength of terms with lower powers has important implications!

As an example we compare the contributions $\sim \frac{9}{4}\overline{\sigma}_0\overline{\lambda}_2$ and $\sim 3\overline{\sigma}_0^3\overline{\lambda}_3$ to the cubic coupling $\gamma_6$ (4.34) which determines the mass split within the scalar octet. If we assume that the second term does not exceed in size the first one we obtain the bound

$$\gamma_6 < \frac{1}{4}\left(18\sigma_0\lambda_2 - \nu\right)\left(\frac{Z_m}{Z_h}\right)^{\frac{3}{2}} \simeq 17.1 \text{ GeV} \qquad (10.1)$$

where we have used $\lambda_2 = 21.3$, $\nu = 6447$ MeV and $Z_h/Z_m = 0.35$ (see Sect. 13). With the help of (2.19) this can be transformed into a bound for the mass difference between the $K_0^*$ and $a_0$ mesons in the scalar octet

$$\overline{M}_{K_0^*}^2 - \overline{M}_{a_o}^2 = 3\gamma_6 v < 0.7 \text{ GeV}^2 \ . \qquad (10.2)$$

This relatively conservative bound seems to disfavor the interpretation of the $a_0(980)$ resonance as a member of the same octet as the $K_0^*(1430)$, since in this case the difference in mass squared would have to exceed 1 GeV². In simpler words, it seems at first sight unlikely that a strange quark mass of about 180 MeV produces a mass difference between strange and non–strange scalar mesons of 450 MeV. Yet, we notice that (10.2) is subject to quark mass corrections from kinetic terms, and we will come back to this issue in Sect. 11.

Before closing this section let us comment on the question if the limiting case $\sigma_0 \to 0$ can be used as an expansion point within a generalized class of linear sigma models. (Of course, the sigma model corresponding to low–energy QCD has a fixed value of $\sigma_0$.) Let us consider an effective quark–meson theory which is supposed to be valid for momentum scales below some cutoff (or compositeness) scale $k_\varphi$. The "classical action" of such a model is parameterized by a potential and, in particular, a mass term $\overline{m}^2(k_\varphi)$. (Quantum fluctuations of modes with momenta $q^2 < k_\varphi^2$ change the form of the effective action and lead to an effective potential as parameterized by (4.22)). The size of $\sigma_0$ can be influenced by the meson mass term $\overline{m}^2(k_\varphi)$ at the scale $k_\varphi$. If the phase transition associated to a variation of $\overline{m}^2(k_\varphi)$ were of second order the order parameter $\sigma_0$ could be arbitrarily small. An expansion in powers of $\sigma_0$ would then always be meaningful for small enough[14] $\sigma_0$. For three flavors the anomaly induces a first order transition and excludes arbitrarily small values of $\sigma_0$. Nevertheless, for small $\sigma_0$ a polynomial expansion of $U(\sigma)$ should be meaningful and we may stop after the term quartic in $\sigma$. In the limit of equal quark masses the potential (4.22) gives

$$U = -3m_g^2\sigma_0^2 - \frac{1}{2}\nu\sigma_0^3 + \frac{9}{2}\lambda_1\sigma_0^4$$
$$+ \left(3m_g^2 + \frac{3}{2}\nu\sigma_0 - 9\lambda_1\sigma_0^2\right)\sigma^2 - \nu\sigma^3 + \frac{9}{2}\lambda_1\sigma^4 \ .$$
$$(10.3)$$

The requirement $U(\sigma_0) < U(0)$ implies a lower bound

$$\sigma_0^2 > \frac{1}{9\lambda_1}\left(\nu\sigma_0 + 6m_g^2\right) \qquad (10.4)$$

whereas the positivity of the mass term at $\sigma_0$ requires

$$\sigma_0^2 > \frac{1}{12\lambda_1}\left(\nu\sigma_0 - 2m_g^2\right) \ . \qquad (10.5)$$

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[14] At nonvanishing temperature $\sigma_0(T)$ typically decreases as $T$ increases. For a second order high temperature phase transition one can always expand such a system for a very small even though perhaps not arbitrarily small value of $\sigma_0$. In the limit $\sigma_0 \to 0$ one may encounter nonanalyticities associated to the critical three dimensional behavior at the transition

On the other hand, the dimensionless coupling $\lambda_1$ is typically bounded from above as a result of the "triviality" of $\Phi^4$–theory. (More precisely, the infrared interval of allowed renormalized quartic couplings is bounded.) Comparing (10.5) with the definition of $x_\sigma$ we find that for vanishing quark masses ($m_g^2 = 0$) the expansion coefficient must obey $x_\sigma > \frac{\lambda_2}{12\lambda_1}$ and can therefore not be arbitrarily small. Despite this caveat there seems to be enough room for a meaningful $\sigma_0$–expansion. It is interesting to note that for given $\sigma_0$ the inequalities (10.4) and (10.5) can also be used to establish lower bounds for $\lambda_1$ which hold for a polynomial approximation (10.3). Taking $\nu$, $m_g^2$ and $\sigma_0$ from Sect. 13 one finds $\lambda_1 > 48.9$ and $\lambda_1 > 1.1$, respectively.

## 11 Mass relations for the scalar octet

In this section we want to determine the masses of the members of the $0^{++}$ octet contained in $\Phi$. Together with the $0^{++}$ singlet these scalars play for spontaneous chiral symmetry breaking the same role as the Higgs scalar in the electroweak theory. Since the isospin singlet member of the octet has the same quantum numbers as a possible scalar glueball the determination of its mass is also important for the identification of glueball state candidates. We restrict most of the discussion to the linear order in an expansion in powers of quark masses and we neglect isospin violation. The quark mass corrections to the kinetic terms for the scalar octet to linear order in $\Delta$ arise from an interaction analogous to (6.1)

$$\mathcal{L}_{\text{kin}}^{(1,s)} = \frac{1}{4}\omega_h \operatorname{Tr} h\partial^\mu h\partial_\mu h . \tag{11.1}$$

Since in the linear sigma model several of the generalized kinetic terms (4.6) contribute to this invariant we will treat $\omega_h v$ as a free parameter to order $\Delta$. In addition, we will consider a particular term contributing to second order in $\Delta$

$$\mathcal{L}_{\text{kin}}^{(2,s)} = \frac{1}{4}\zeta_h \operatorname{Tr} \{(h\partial^\mu h - \partial^\mu hh)(h\partial_\mu h - \partial_\mu hh)\} . \tag{11.2}$$

This term is induced by the exchange of the scalar state contained in the divergence of the vector meson field $\partial_\mu \rho_V^\mu$ (cf. Appendix B) with a sizeable coefficient $\zeta_h v^2 \gtrsim 0.02$. Combining (11.1) and (11.2) and neglecting terms $\sim \omega_h \zeta_h v^3$ this leads to different wave function renormalizations for $a_0$, $K^*$ and $f_8$

$$\begin{aligned} Z_{a_o} &= 1 - \omega_h v \\ Z_{K_o^*} &= \left(1 + \tfrac{1}{2}\omega_h v\right)\left(1 - 9\zeta_h v^2\right) \\ Z_{f8} &= 1 + \omega_h v . \end{aligned} \tag{11.3}$$

We next have to include the effects of higher derivative terms using the definition of the wave function renormalization constants (6.11). The discussion is completely analogous to Sect. 6 and the dominant higher derivative terms lead to a replacement of $\omega_h$ by $\overline{\omega}_h = \omega_h + 2\Delta Z_{K_o^*}/v$. (Here we have adopted a definition of $Z_h$ such that $2Z_{K_o^*} +$

$Z_{a_o} = 1$ for $\zeta_h = 0$.) We note that the effects from a nonvanishing $\zeta_h$ can be absorbed in an effective mass term

$$\hat{M}_{K_o^*}^2 = M_{K_o^*}^2 \left(1 - 9\zeta_h v^2\right) . \tag{11.4}$$

Up to the replacement of the physical mass $M_{K_o^*}^2$ by the $\zeta_h$ dependent quantity $\hat{M}_{K_o^*}^2$ our discussion systematically only includes terms linear[15] in $\Delta$.

From (5.8) we now obtain

$$\begin{aligned} m_h^2 &= \frac{1}{3}\left[2\overline{M}_{K_o^*}^2 + \overline{M}_{a_o}^2\right] \\ &= \frac{1}{3}\left[2\hat{M}_{K_o^*}^2 + M_{a_o}^2 + \overline{\omega}_h v\left(\hat{M}_{K_o^*}^2 - M_{a_o}^2\right)\right]. \end{aligned} \tag{11.5}$$

Hence, the linear quark mass corrections to the kinetic terms $\sim \overline{\omega}_h$ modify $m_h^2$ only to quadratic order in $\Delta$. For $M_{a_o} = 1320(983)$ MeV, $\overline{\omega}_h = 0$ and $\zeta_h = 0$ we find[16] $m_h \simeq 1394(1298)$ MeV. The dominant correction to the lowest order relation is most likely due to the term $\sim \zeta_h$. For $\zeta_h v^2 = 0.02$ one obtains $m_h = 1303(1200)$ MeV whereas $\zeta_h v^2 = 0.04$ yields already a large shift to $m_h = 1206(1093)$ MeV. We are now in a position to compute $Z_h/Z_m$ from (5.16). For $\overline{\omega}_m v = -0.20$ we obtain

$$\frac{Z_h}{Z_m} \simeq 0.39(0.45) , \quad 0.45(0.53) , \quad 0.53(0.64) \tag{11.6}$$

where the three values correspond to $\zeta_h v^2 = 0, 0.02, 0.04$. Comparing with (5.17) we see that the quark mass corrections to the kinetic terms strongly influence the determination of $Z_h/Z_m$. We conclude that $1 - \frac{Z_h}{Z_m}$ can not be treated as a very small number. The difference between $Z_h$ and $Z_m$ has to be included for any systematic discussion of the scalars within the linear sigma model! The neglect of this difference in earlier works [7]–[14] partially explains the quantitative differences with our results.

We are now ready to reexamine the mass splitting in the scalar octet (10.2). Including the corrections arising from (11.3) we find

$$\hat{M}_{K_o^*}^2 - M_{a_o}^2 = \frac{3\gamma_6 v - \frac{3}{2}m_h^2\overline{\omega}_h v}{1 - \frac{1}{2}\overline{\omega}_h v - \frac{1}{2}(\overline{\omega}_h v)^2} . \tag{11.7}$$

One sees that large negative values of $\overline{\omega}_h v$ can considerably increase the mass difference between the $K_0^*$ and the $a_0$ for given $\gamma_6 v$. The same holds for $\zeta_h v^2 > 0$. This weakens the argument of the preceding section against the association of the $a_0$ meson with the resonance $a_0(980)$. In Fig. 5 we plot $M_{a_o}$ as a function of $\overline{\omega}_h v$ for $Z_p = Z_m$ and three different values of $\zeta_h v^2$, with bands corresponding to ranges of $\lambda_3$ between $\sigma_0^2\lambda_3 = -\lambda_2/4$ (lower curves) and

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[15] The exception from a systematic procedure of keeping only terms in the effective action for scalars that contribute to linear order in $\Delta$ is motivated by the well identified mechanism that induces a sizeable $\zeta_h v^2$ (cf. Appendix B)

[16] A similar observation holds in the pseudoscalar sector for $m_g^2$ and we obtain $m_g \simeq 393$ MeV
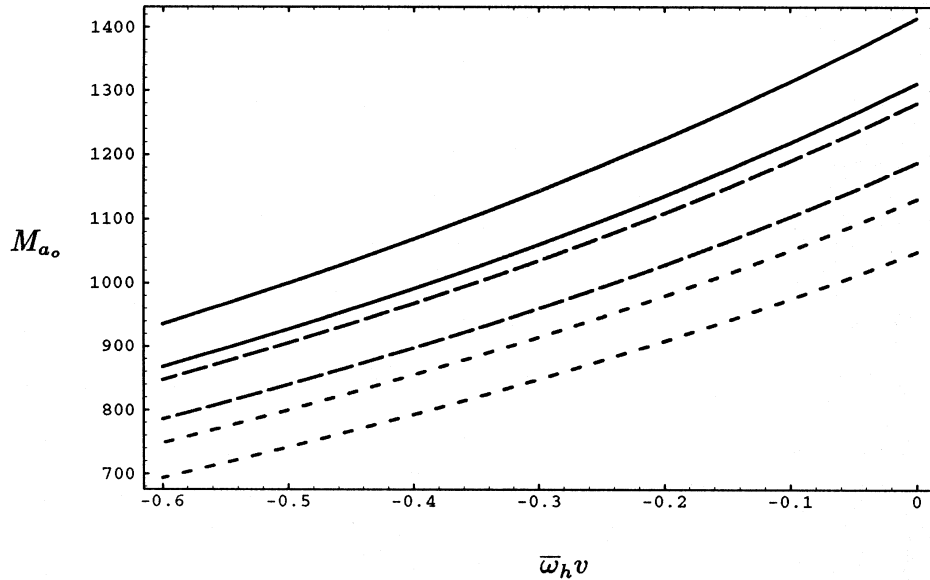
**Fig. 5.** The plot shows $M_{a_o}$ as a function of $\overline{\omega}_h v$ for $Z_p/Z_m = 0.9$, $\omega_m v = \overline{\omega}_m v = -0.20$ and three values of $\zeta_h v^2 = 0$ (*solid lines*), 0.02 (*dashed lines*) and 0.04 (*dotted lines*). The upper curve in each band corresponds to $\sigma_0^2 \lambda_3 = -\lambda_2/4$ and the lower one to $\sigma_0^2 \lambda_3 = \lambda_2/4$

$\sigma_0^2 \lambda_3 = \lambda_2/4$ (upper curves). Here we have used (5.18) together with

$$\frac{\gamma_6}{\gamma_2} = \frac{1}{2} \frac{\hat{M}_{K_o^*}^2 - M_{a_o}^2 + \overline{\omega}_h v \left(\frac{1}{2}\hat{M}_{K_o^*}^2 + M_{a_o}^2\right)}{M_K^2 - M_\pi^2 + \overline{\omega}_m v \left(\frac{1}{2}M_K^2 + M_\pi^2\right)} . \quad (11.8)$$

In fact, large quark mass corrections to the scalar kinetic terms seem the only plausible possibility for the choice of the $a_0(980)$. This would indicate a large mixing between two–kaon states and the $a_0(980)$ (see Appendix C). Consequently, it could explain why the $a_0(980)$ behaves in many respects similarly to a $\overline{q}q\overline{q}q$ state even though it may belong to an octet of $\overline{q}q$ states. We should also point out that for large values of $|\overline{\omega}_h v|$ the quark mass expansion becomes questionable in the scalar sector. From the linearized expressions

$$M_{a_o}^2 = m_h^2 - 2\left(\gamma_6 v - \frac{1}{2}m_h^2 \overline{\omega}_h v\right)$$

$$\hat{M}_{K_o^*}^2 = m_h^2 + \left(\gamma_6 v - \frac{1}{2}m_h^2 \overline{\omega}_h v\right) \quad (11.9)$$

we infer the ratio of the first order correction for $M_{a_o}^2$ as compared to $m_h^2$

$$\frac{2\left(\gamma_6 v - \frac{1}{2}m_h^2 \overline{\omega}_h v\right)}{m_h^2} = \frac{2}{3}\frac{\hat{M}_{K_o^*}^2 - M_{a_o}^2}{m_h^2} \quad (11.10)$$

$$\simeq 0.10(0.43), \ -0.03(0.33),$$
$$-0.20(0.19)$$

for $\zeta_h v^2 = 0, 0.02, 0.04$. Apparently, for $\zeta_h v^2 = 0$ a good convergence of the quark mass expansion is only realized for the assignment $a_0(1320)$. For larger values of $\zeta_h v^2$

as infered from the leading mixing approximation in Appendix B a reasonable convergence can also be obtained for $a_0(980)$.

Summarizing the various aspects of the problem of the correct assignment of the isotriplet belonging to the same octet as the $K_0^*(1430)$ we may state that the association $a_0(1320)$ would make the understanding easier only in case of a standard kinetic term for $h$. Taking into account non-minimal kinetic terms there is no conclusive argument to rule out the $a_0(980)$ as a member of the scalar octet. For the latter assignment one expects important mixing effects with two–kaon states. Actually, such large mixing effects concern presumably only the $a_0$ and not the other members of the scalar octet. It may therefore be preferable not to include these mixing effects into the parameter $\omega_h v$ appearing in (11.3) but to treat them as additional corrections to the $a_0$ propagator only. In this case the size of $\omega_h v$ is expected to remain small, $|\omega_h v| \lesssim 0.1$, but the physical mass of the $a_0$ is related to $\overline{M}_{a_0}$ by an unknown factor reflecting the mixing. The value of $M_{a_o}$ appearing in formulae like (11.5) or (11.11) below should then be replaced by an effective mass $\hat{M}_{a_o}$ somewhat above 1 GeV (say around 1100 MeV). A natural mechanism of "threshold mass shifting" leading to a physical mass $M_{a_o} = 980$ MeV is described in Appendix C.

We finally want to show that the Gell-Mann–Okubo type mass relation (2.22) is not affected by linear quark mass corrections to the kinetic terms. Inserting $M_{a_o}^2 = \overline{M}_{a_o}^2 Z_{a_o}^{-1}$, $M_{K_o^*}^2 = \overline{M}_{K_o^*}^2 Z_{K_o^*}^{-1}$, $M_{f_8}^2 = \overline{M}_{f_8}^2 Z_{f_8}^{-1}$ into (2.22) one obtains to linear order in $\overline{\omega}_h v$ the relation

$$M_{f_8}^2 = \frac{4}{3}\hat{M}_{K_o^*}^2 - \frac{1}{3}M_{a_o}^2 - \frac{2}{3}\overline{\omega}_h v\left(\hat{M}_{K_o^*}^2 - M_{a_o}^2\right) . \quad (11.11)$$

The correction to the relation (2.22) is indeed only quadratic in $\Delta$. For given $M_{a_o} = 1320$ MeV and $M_{K_o^*} = 1430$ MeV
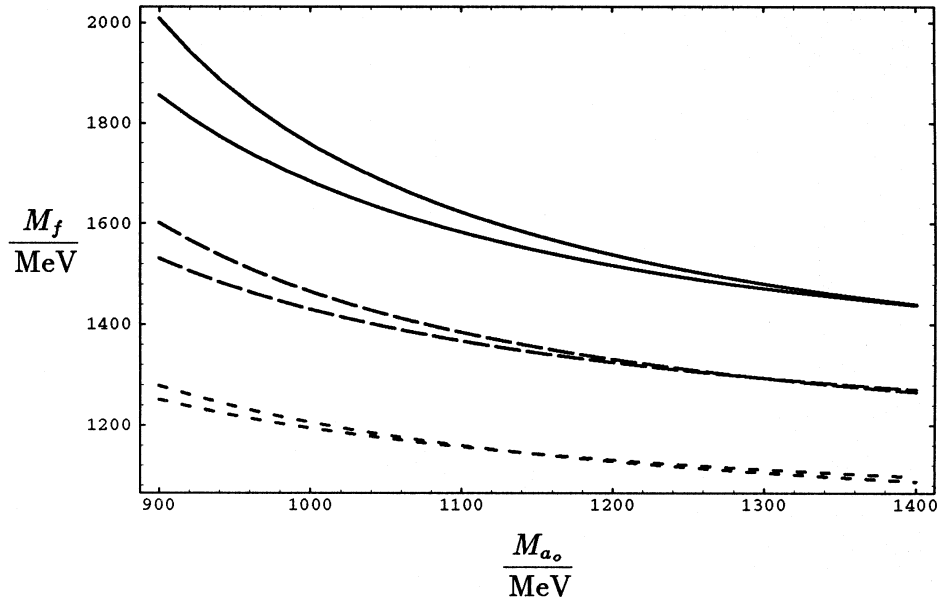
**Fig. 6.** The plot shows $M_f$ as a function of $M_{a_o}$ according to the scalar Gell-Mann–Okubo relation (2.22) with (11.3), $Z_p/Z_m = 0.9$, $\omega_m v = \overline{\omega}_m v = -0.20$ and fixed $M_{K_o^*} = 1430$ MeV. The bands correspond to values of $\lambda_3 \sigma_0^2$ between $-\lambda_2/4$ (*upper curves*) and $\lambda_2/4$ (*lower curves*) and we give results for $\zeta_h v^2 = 0$ (*solid lines*), 0.02 (*dashed lines*) and 0.04 (*dotted lines*)

the symmetry relation (11.11) yields for $\zeta_h v^2 = 0$

$$M_{f_8} \simeq 1465 \text{ MeV} . \qquad (11.12)$$

On the other hand, for $\zeta_h v^2 = 0.02$ and $\hat{M}_{a_o} = 1100$ MeV we find

$$M_{f_8} \simeq 1354 \text{ MeV} . \qquad (11.13)$$

Taking into account the uncertainties from the mixing with the scalar singlet $s$ both values are consistent with the observed broad resonance[17] $f_0(1300)$. In Fig. 6 we have plotted $M_{f_8}$ as a function[18] of $\hat{M}_{a_o}$ in order to demonstrate the relative insensitivity of $M_{f_8}$ on the identification of the $a_0$ meson. For this purpose we have used $\zeta_h = 0, 0.02$ and 0.04 and $\lambda_3 \sigma_0^2 = \pm \lambda_2/4$. For each set of parameters we have determined $\overline{\omega}_h v$ from Fig. 5. Figure 6 demonstrates that for $\zeta_h v^2 \gtrsim 0.02$ values of $M_{f_8}$ below 1500 MeV are preferred. On the other hand, a value of $M_{f8}$ below 1100 MeV would require a very substantial mixing of $K_0^*$ with a state in $\partial_\mu \rho_V^\mu$ ($\zeta_h v^2 \gtrsim 0.04$.). The identification of either the $f_0(1590)$ or the $f_0(980)$ with the octet seems therefore disfavored.

In this context it is perhaps interesting to note that the branching ratio [15]

$$R = \frac{\Gamma\left(f_0(1300) \to 2\pi\right)}{\Gamma\left(f_0(1300) \to K\overline{K}\right)} \simeq 12.5 \qquad (11.14)$$

is consistent with an octet assignment of the $f_0(1300)$. For a pure octet this ratio should be around 6 and a relatively

small admixture of a singlet state could easily explain a further enhancement. (Whereas a pure singlet state would lead to a much smaller ratio $R \simeq 1.5$ a second solution for large $R$ corresponds to a large octet–singlet mixing angle, i.e. $\tan \vartheta_s \simeq -0.9$ for equal $sm^2$ and $hm^2$ couplings.) On the other hand, the branching ratios of the higher mass resonance $f_0(1590)$ seem compatible with a singlet with large cubic coupling $\sim p^2 s$, but not with an octet. For the $f_0(980)$ an assignment is difficult in view of the presumably large mixing with two–kaon states.

In summary, two natural scenarios for the scalar nonet seem to be compatible with the parameters of the linear meson model extracted from the pseudoscalar sector: For one scenario the isotriplet corresponds to the $a_0(980)$ and the singlet (or dominant $\overline{s}s$ state in case of large $|\vartheta_s|$) is associated with the $f_0(980)$. These four states are largely influenced by mixing with two–kaon states. The other $f_0$ state of the nonet (dominantly an octet state in case of small $|\vartheta_s|$) corresponds to the $f_0(1300)$. In this case a relatively large mixing in the strange sector with $\partial_\mu \rho_V^\mu$ (large $\zeta_h v^2$) explains why the $K_0^*(1430)$ has the highest mass in the nonet. The other scenario has a larger average mass $m_h$ of the octet. The triplet is associated with the proposed $a_0(1320)$ which is not far below the doublets $K_0^*(1430)$. The $a_0(980)$ and $f_0(980)$ are dominantly four–quark states or $K\overline{K}$ molecules in this case. Again, the octet state is the $f_0(1300)$. The singlet corresponds[19] either to $f_0(1590)$ or its width is too large to be detected. Reliable information about the value of $\lambda_1$ would certainly be of great interest for further pinning down the possible options.

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[17] The mass of the $f_0(1300)$ is not determined very precisely. It could easily be around 1400 MeV

[18] In Figs. 5 and 6 we have not distinguished between $M_{a_o}$ and $\hat{M}_{a_o}$. Taking into account the additional mixing with two–kaon states for the $a_0$ the relevant axis actually shows $\hat{M}_{a_o}$

[19] There are other not so well established resonances $f_0(1510)$ and $f_0(1525)$ [19] which may be identical with the $f_0(1590)$ or also be possible candidates for the singlet
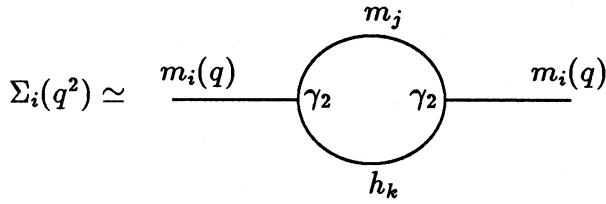
$$\Sigma_i(q^2) \simeq \quad \underset{m_i(q)}{\rule{0pt}{0pt}} \overset{m_j}{\underset{h_k}{\gamma_2 \quad \gamma_2}} \underset{m_i(q)}{\rule{0pt}{0pt}}$$

**Fig. 7.** One-loop diagram for the dominant contribution to $\Sigma_i(q^2)$

## 12 Higher derivative contributions

In this section we investigate deviations of the meson propagator from the approximated form $G_i = (Z_i q^2 + \overline{M}_i^2)^{-1}$. We will first concentrate on the flavored pseudoscalars. In the language of Sect. 6 we want to make an estimate of the corrections $\Delta Z_i$ (6.13). Within a systematic quark mass expansion we need to order $\Delta^2$ only the $q^4$ correction to the inverse propagator in an approximation where it is independent of the quark masses. This correction arises from a term involving four derivatives

$$\mathcal{L}_{\mathrm{kin}(4)} = \frac{\overline{H}_m}{4} \, \mathrm{Tr}\left( \partial^2 m \partial^2 m - q_0^2 \partial^\mu m \partial_\mu m \right) \qquad (12.1)$$

where $q_0^2$ is chosen according to (6.19). It involves one additional parameter $\overline{H}_m$ which determines the ratio

$$\frac{\overline{\omega}_m}{\omega_m} = 1 + \delta_\omega = 1 - \frac{2}{3}\overline{H}_m \frac{M_K^2 - M_\pi^2}{\omega_m v} \, . \qquad (12.2)$$

As a first observation we notice that $\overline{H}_m$ receives contributions from the mixing of the pseudoscalar mesons with other states. We infer from Appendix C that the mixing with other $0^{-+}$ octets indeed induces higher derivative corrections because of the momentum dependence of the propagators of the additional states that are integrated out. If we assume that the dominant contribution to $\omega_m$ arises through mixing with other states we can identify in (C.7) $\omega_m$ with $\omega_m^{(\rho)}$ and find

$$\delta_\omega^{(\rho)} = \frac{1}{6} \frac{(M_K^2 - M_\pi^2)}{(M_P^2 - m_m^2)} \frac{[2\overline{f}_K + \overline{f}_\pi]}{[\overline{f}_K - \overline{f}_\pi]} \simeq 0.07(0.05) \qquad (12.3)$$

where we used $M_P = 2280$ MeV(2670 MeV) (cf. Appendix C and Table 9 in Appendix B). One may also estimate $K_8$ according to (6.19) and finds

$$K_8^{(\rho)} \simeq 0.002(0.001) \, . \qquad (12.4)$$

This is indeed negligible for the wave function renormalizations as compared to $\overline{\omega}_m v$.

A different contribution to the higher derivative term (12.1) arises from loops of meson fluctuations. For an estimate of their importance we use the modified loop expansion of [16] ("systematically resummed perturbation theory"). This allows to compute the deviations of the inverse propagator from the form $q^2 + M_i^2$ in terms of $M_i^2$ and effective $1PI$ cubic vertices. It is crucial in our context that instead of "classical vertices" only the $1PI$ Green

functions appear in the perturbative series since only the latter are directly calculable from the present phenomenological analysis. Also the loop expansion is only used for "higher order couplings" where it converges reasonably well. (We do not expect a good convergence for quantities like $Z_h/Z_m$ etc.) Let us write the inverse propagator for a member of the pseudoscalar octet as

$$G_i^{-1}(q^2) = q^2 + M_i^2 + \tilde{\Sigma}_i(q^2) \, . \qquad (12.5)$$

Here we have subtracted from the usual self energy $\Sigma_i(q^2)$ those parts which are already contained in the effective wave function renormalizations $\overline{Z}_i$ and mass terms $\overline{M}_i^2$

$$\tilde{\Sigma}_i(q^2) = \Sigma_i(q^2) - \Sigma_i(0) - \frac{q^2}{q_0^2} \left( \Sigma_i(q_0^2) - \Sigma_i(0) \right) \, . \quad (12.6)$$

This definition implies

$$\tilde{\Sigma}_i(0) = \tilde{\Sigma}_i(q_0^2) = 0 \qquad (12.7)$$

and we use $q_0^2 = -m_m^2$. In consequence, $\tilde{\Sigma}_i(q^2)$ contains only contributions to higher derivative terms. The dominant one–loop contribution to $\Sigma_m(q^2)$ for the pseudoscalar octet is depicted in Fig. 7. It involves the propagation of a scalar and a pseudoscalar in the loop and we therefore need the cubic coupling $\gamma_2$ (cf. Sect. 2). We observe that the subtraction (12.6) makes the usual one–loop expression ultraviolet finite. In fact, if the momentum dependence of the effective three–point vertex is not too strong the momentum integral for the difference $\Sigma_m(q^2) - \Sigma_m(0)$ is dominated by momenta in the range between the masses of the two particles propagating in the loop. We will use here the approximation of a constant cubic vertex which we approximate by its value to lowest order in the quark mass expansion given by $\gamma_2$. We also neglect the octet mass splitting for the particles propagating in the loop for which we use average masses $m_m = 412$ MeV and $m_h = 1394$ MeV. To lowest order in the quark mass expansion we are interested in the effective coupling $\overline{H}_m$ (6.12) which is given by

$$\overline{H}_m = \frac{\partial}{\partial q^2} \left( \frac{\tilde{\Sigma}_m}{q^2} \right)_{q^2 = q_0^2} \, . \qquad (12.8)$$

We are interested in the momentum range $-q^2 < (m_h - m_m)^2$ for which the one–loop contribution is given by

$$\begin{aligned}
&\Sigma_m^{(1)}(q^2) - \Sigma_m^{(1)}(0) \\
&\simeq \frac{5\gamma_2^2}{48\pi^2} \left\{ 1 - \left[ \frac{m_m^2 - m_h^2}{q^2} + \frac{m_m^2 + m_h^2}{m_m^2 - m_h^2} \right] \ln \frac{m_h}{m_m} \right. \\
&\quad - \frac{1}{q^2} \sqrt{(m_m + m_h)^2 + q^2}\sqrt{(m_m - m_h)^2 + q^2} \\
&\quad \left. \times \ln \frac{\sqrt{(m_m + m_h)^2 + q^2} + \sqrt{(m_m - m_h)^2 + q^2}}{\sqrt{(m_m + m_h)^2 + q^2} - \sqrt{(m_m - m_h)^2 + q^2}} \right\} \, .
\end{aligned}$$
$$(12.9)$$

Here we have neglected contributions $\sim \gamma_1^2, \gamma_3^2$ as well as the $\eta$–$\eta'$ mixing. For $q_0^2 = -m_m^2$ this yields the one–loop contribution to $\overline{H}_m$

**Table 1.** This table shows the phenomenological input used in this work. All values are given in MeV. The charged meson masses are electromagnetically corrected

| $M_{\pi\pm}$ | $M_{K\pm}$ | $M_{K^0}$ | $M_{\eta'}$ | $f_\pi$ | $f_{K\pm}$ |
|---|---|---|---|---|---|
| 135.1 | 492.4 | 497.7 | 957.8 | 92.4 | 113.0 |

$$\overline{H}_m^{(1)} \simeq -\frac{5}{48\pi^2} \frac{\gamma_2^2}{m_m^4}$$

$$\times \left\{ 2 - \frac{2m_h^4 - 5m_h^2 m_m^2 + m_m^4}{m_m^2(m_h^2 - m_m^2)} \ln \frac{m_h}{m_m} \right. \quad (12.10)$$

$$+ m_h \sqrt{m_h^2 - 4m_m^2} \left( \frac{2}{m_m^2} + \frac{1}{m_h^2 - 4m_m^2} \right)$$

$$\left. \times \ln \frac{\sqrt{m_h + 2m_m} + \sqrt{m_h - 2m_m}}{\sqrt{m_h + 2m_m} - \sqrt{m_h - 2m_m}} \right\}.$$

Using $\gamma_2 \simeq 8000$ MeV and $\omega_m v \simeq -0.20$ (see Sect. 13) this results in

$$\overline{H}_m^{(1)} \simeq 1.64 \cdot 10^{-8} \text{ MeV}^{-2} \qquad (12.11)$$

or

$$\delta_\omega^{(1)} \simeq 0.012 . \qquad (12.12)$$

We see that loop corrections to the higher derivative terms are negligible as compared to contributions arising through the mixing with other states. We will therefore assume that the higher derivative terms are dominated by such mixings and estimate

$$\frac{\omega_m}{\overline{\omega}_m} = \frac{1}{1 + \delta_\omega^{(\rho)}} \simeq 0.95 . \qquad (12.13)$$

Even though $(\overline{\omega}_m - \omega_m)/\omega_m$ is formally not suppressed by powers of $\Delta$ we see that this higher derivative effect is actually small.

In the $\eta$–$\eta'$ sector we need information about the momentum dependence of $z_8(q^2)$, $z_p(q^2)$ and $\hat{\omega}(q^2)$ (6.26). The relevant quantities are $d_8 = z_8(-M_{\eta'}^2) - 1$, $d_p = z_p(-M_{\eta'}^2) - 1$ and $d_\omega = (\hat{\omega}(-M_{\eta'}^2) - \hat{\omega})/\hat{\omega}$ where we remind the reader of the definitions $z_8(-M_\eta^2) = z_p(-M_\eta^2) = 1$ and $\hat{\omega}(-M_\eta^2) = \hat{\omega}$. Since the mass difference $M_{\eta'}^2 - M_\eta^2$ exceeds substantially $M_K^2 - M_\pi^2$ the higher derivative corrections in the $\eta$–$\eta'$ sector could be somewhat larger than for the flavored mesons. Altogether, the contributions beyond the lowest order in the derivative expansion involve four additional dimensionless parameters, $\delta_\omega$, $d_8$, $d_p$ and $d_\omega$. Their absolute size is expected to be small if the derivative expansion converges. Since the predictions of the lowest order in the derivative expansion come already very close to the experimental values of $f_\eta$ and $f_{\eta'}$ it seems not difficult to achieve agreement with observation by using small but otherwise arbitrary values for these four parameters. The expansion in powers of $\sigma_0$ may lead to some approximate relations between $\delta_\omega$, $d_8$, $d_p$ and $d_\omega$ but we will not pursue this issue here further.

**Table 2.** The table shows the four different combinations of the parameters $\omega_m v$, $Z_p/Z_m$ and $\omega_m/\overline{\omega}_m$ used in Sect. 13. Line (d) corresponds to the leading mixing approximation for which $\omega_m v$ and $\omega_m/\overline{\omega}_m$ are computed and do therefore not play the role of free input parameters

|  | $\omega_m v$ | $Z_p/Z_m$ | $\omega_m/\overline{\omega}_m$ | $\overline{\omega}_m v$ |
|---|---|---|---|---|
| (a) | −0.22 | 1.0 | 1.0 | −0.22 |
| (b) | −0.20 | 0.9 | 1.0 | −0.20 |
| (c) | −0.22 | 0.9 | 0.9 | −0.24 |
| (d) | −0.17 | 0.9 | 0.95 | −0.18 |

**Table 3.** Values for wave function renormalizations $Z_i$ and zero momentum mass parameters $\overline{M}_i$

|  | $Z_\pi$ | $Z_{K\pm}$ | $Z_{K\pm} - Z_{K^0}$ | $\frac{\overline{M}_{\pi\pm}}{\text{MeV}}$ | $\frac{\overline{M}_{K\pm}}{\text{MeV}}$ | $\frac{\overline{M}_{K^0}}{\text{MeV}}$ |
|---|---|---|---|---|---|---|
| (a) | 1.22 | 0.89 | 0.0063 | 149.2 | 465.3 | 468.7 |
| (b) | 1.20 | 0.90 | 0.0058 | 148.0 | 467.9 | 471.4 |
| (c) | 1.24 | 0.88 | 0.0068 | 150.7 | 462.2 | 465.4 |
| (d) | 1.18 | 0.91 | 0.0052 | 146.5 | 470.9 | 474.6 |

**Table 4.** Expectation values of scalar singlet and octets

|  | $\frac{\overline{f}_\pi}{\text{MeV}}$ | $\frac{\overline{f}_{K\pm}}{\text{MeV}}$ | $\frac{\sigma_0}{\text{MeV}}$ | $\left(\frac{Z_m}{Z_h}\right)^{1/2} \frac{v}{\text{MeV}}$ | | $\left(\frac{Z_m}{Z_h}\right)^{1/2} \frac{w}{\text{MeV}}$ |
|---|---|---|---|---|---|---|
| (a) | 83.7 | 119.6 | 53.9 | 24.2 | | −0.69 |
| (b) | 84.3 | 118.9 | 53.8 | 23.3 | | −0.67 |
| (c) | 82.8 | 120.4 | 54.1 | 25.3 | | −0.71 |
| (d) | 85.2 | 118.2 | 53.7 | 22.2 | | −0.66 |

Instead, we conclude this section by a description of the predictions of the "leading mixing approximation". For this purpose we assume that all corrections to the kinetic terms — both, quark mass and higher derivative corrections — are due to a mixing with higher states contained in the divergence of the axialvector field $\partial_\mu \rho_A^\mu$. The formalism is described in appendices B and C. All parameters appearing in the kinetic terms can be computed in terms of masses and couplings of the vector– and axialvector fields. Most of these couplings can be determined from observation (cf. Appendix B). There remains essentially only one important free parameter $Z_P$ which appears in the term $(1/4) \text{Tr} \left\{ \tilde{Z}_P (\partial_\mu \rho_V^\mu)^2 + Z_P (\partial_\mu \rho_A^\mu)^2 \right\}$. This parameter determines the strength of the higher derivative terms induced by the mixing with $M_P^2 \sim Z_P^{-1}$ in (12.3). In Fig. 8a we plot the values of $M_\eta/M_\eta^{\text{exp}}$, $f_\eta/f_\eta^{\text{exp}}$ and $f_{\eta'}/f_{\eta'}^{\text{exp}}$ as functions of $Z_P$. (The higher derivative corrections vanish for $Z_P = 0$.) For this plot we use $x_\rho = 1$ (cf. Appendix B) and employ the leading mixing approximation which assumes $\omega_m = \omega_m^{(\rho)}$ and $\overline{\omega}_m/\omega_m$ given by (12.3), (12.12). The higher derivative contributions in (6.26) and (A.26) are now included according to (C.9), (C.13) and (C.14). We see that a reasonable picture emerges for $Z_P \simeq 0.22$. It is consistent with the assumption that the nonminimal kinetic terms for the pseudoscalars are dominated by the mixing with $\partial_\mu \rho_A^\mu$ or the "partial Higgs effect". The remaining differences of the curves from
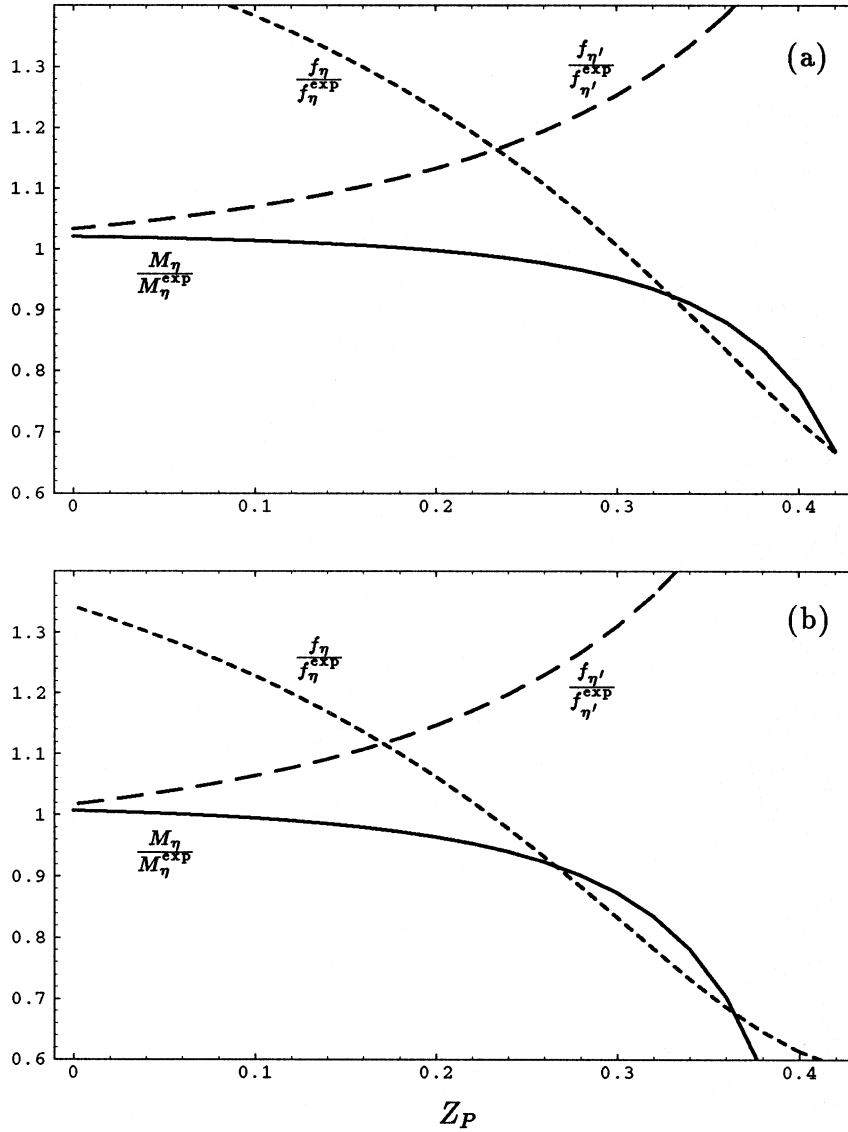
**Fig. 8.** The plots show the curves for $M_\eta/M_\eta^{\mathrm{exp}}$ (*solid line*), $f_\eta/f_\eta^{\mathrm{exp}}$ (*dotted line*) and $f_{\eta'}/f_{\eta'}^{\mathrm{exp}}$ (*dashed line*) in the "leading mixing approximation" as functions of $Z_P$ for $x_\rho = 1$, $Z_p/Z_m = 1$ **a** and $Z_p/Z_m = 0.9$ **b**

one can reasonably be attributed to subleading effects beyond the leading mixing approximation, as described in the more general framework of the main text. For a demonstration we also show in Fig. 8b the situation which arises if in addition to the contributions from the leading mixing approximation one also includes a nonvanishing $U_\varphi$ in the kinetic terms (4.6). For this plot we have chosen $U_\varphi$ such that $Z_p/Z_m = 0.9$ (cf. (8.10)). The agreement with observation improves and the optimal value of $Z_P$ is shifted to somewhat smaller values. We emphasize that the leading mixing approximation is complementary to the formal expansion in powers of quark masses. It is encouraging that a simple mechanism (the partial Higgs effect) can apparently explain the dominant parts of the parameters appearing in the systematic quark mass expansion.

## 13 Results

For the convenience of the reader we summarize in this section the results of a numerical solution of our equations. We observe that the pseudoscalar sector can be treated independently from the scalar sector. For the flavored pseudoscalars there are two small parameters whose influence is rather modest, namely $Z_p/Z_m - 1$ and $\delta_\omega = \overline{\omega}_m/\omega_m - 1$. Three additional small parameters $d_\omega$, $d_8$, $d_p$ characterize the most general form of the higher derivative terms in the $\eta$–$\eta'$ sector. For the first two lines $(a)$ and $(b)$ in our tables we use the first order in the derivative expansion, i.e. $\delta_\omega = d_\omega = d_8 = d_p = 0$. We present two values $Z_p/Z_m = 1.0$ and $0.9$. The value of the dominant free parameter $\omega_m v$ is chosen such that $M_\eta$ comes out close to its experimental value (cf. Fig. 1). Going beyond the first order in the derivative expansion we include in line $(c)$ the higher derivative corrections corresponding to a

**Table 5.** Parameters of the linear meson model

| | $\frac{m_g^2}{\text{MeV}^2}$ | $\frac{\nu}{\text{MeV}}$ | $\lambda_2$ | $\hat{\omega}$ | $\left(\frac{Z_h}{Z_m}\right)^{1/2}$ | $\frac{\gamma_2}{\text{MeV}}$ | $\frac{\gamma_2 v}{\text{MeV}^2}$ | $\frac{\gamma_2 w}{\text{MeV}^2}$ | $\frac{\gamma_3 v}{\text{MeV}^2}$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| (a) | $(390.9)^2$ | 6814 | 17.0 | 0.35 | 4780 | $(340.1)^2$ | $-(57.4)^2$ | $-(80.9)^2$ |
| (b) | $(392.9)^2$ | 6447 | 21.3 | 0.30 | 4942 | $(339.3)^2$ | $-(57.7)^2$ | $(46.1)^2$ |
| (c) | $(388.6)^2$ | 5964 | 17.5 | 0.33 | 4405 | $(333.8)^2$ | $-(55.7)^2$ | $-(38.6)^2$ |
| (d) | $(395.2)^2$ | 6011 | 26.8 | 0.25 | 5167 | $(338.8)^2$ | $-(58.2)^2$ | $(112.3)^2$ |

**Table 6.** "Predictions" for $M_\eta$, $f_\eta$ and $f_{\eta'}$

| | $\frac{M_\eta}{\text{MeV}}$ | $\frac{f_\eta}{\text{MeV}}$ | $\frac{f_{\eta'}}{\text{MeV}}$ | $\frac{f_\eta}{f_\pi}$ | $\frac{f_{\eta'}}{f_\pi}$ | $\frac{f_\eta}{f_\eta^{\text{exp}}}$ | $\frac{f_{\eta'}}{f_{\eta'}^{\text{exp}}}$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| (a) | 550.8 | 106.6 | 93.5 | 1.15 | 1.01 | 1.09 | 1.25 |
| (b) | 546.9 | 113.4 | 83.8 | 1.23 | 0.91 | 1.16 | 1.12 |
| (c) | 549.1 | 103.5 | 88.8 | 1.12 | 0.96 | 1.06 | 1.19 |
| (d) | 536.6 | 111.3 | 82.9 | 1.20 | 0.90 | 1.14 | 1.11 |

nonvanishing $\delta_\omega$. It is taken according to the leading mixing estimate (12.13) such that $\omega_m/\overline{\omega}_m = 0.9$. Finally, we present the results of the leading mixing approximation in line $(d)$. Here all quark mass corrections to the kinetic and all higher derivative terms are determined from the simple assumption that they are induced by the exchange of the axialvector field $\partial_\mu \rho_A^\mu$. Line $(d)$ corresponds to Fig. 8b with $Z_P \simeq 0.16$. In the leading mixing approximation $\omega_m v$ and $\omega_m/\overline{\omega}_m$ are not anymore free parameters and can therefore not be adapted to fix $M_\eta$ to its experimental value. For line $(d)$ the values of $M_\eta$, $f_\eta$ and $f_{\eta'}$ differ somewhat from those obtained for the optimal value $\omega_m v = -0.20$. The leading mixing approximation comes nevertheless quite close to the experimental results. In Table 1 we give our input values for the pseudoscalar sector. Table 2 shows the four different combinations of the parameters $\omega_m v$, $Z_p/Z_m$ and $\omega_m/\overline{\omega}_m$ used for our numerical analysis. The numbers given in the text of this paper correspond to line $(b)$ which may be considered as our best values to first order in the derivative expansion.

The first step is to solve (6.16) for the $Z$–factors and to extract the values of the zero momentum parameters $\overline{M}_i^2$, $\overline{f}_i$. The corresponding relations are (2.6), (5.6), (6.3), (6.10) and the results are found in Tables 3 and 4. We should point out the sizeable differences between $M_i^2$ and $\overline{M}_i^2$ and similarly for the decay constants. They are due to the large value of $|\omega_m v|$. In particular, the difference $\overline{f}_K - \overline{f}_\pi$ is almost twice the value of $f_K - f_\pi$! In the next step we determine in Table 5 the parameters of the linear meson model from (5.2)–(5.4), (5.6), (8.12) and (2.10), (4.29). We observe that the infered value of the cubic coupling $\nu$ depends only moderately on the details of the effective meson model. In contrast, the uncertainty for the quartic coupling $\lambda_2$ remains substantial. Table 5 also contains information about the cubic coupling $\gamma_2$ between two pseudoscalar octets and the scalar octet as well as for the coupling $\gamma_3$ between the pseudoscalar octet, the pseudoscalar singlet and the scalar octet. Even though the sign of $\gamma_3$ remains undetermined we find $|\gamma_3/\gamma_2| \lesssim 0.1$.

The coupling $\gamma_2$ therefore dominates the decay of the $0^{++}$ mesons.

We are now ready to compute the mass matrix elements of the $\eta$–$\eta'$ sector using (5.4). The eigenvalues $M_\eta$, $M_{\eta'}$ follow from (6.29) the mixing angles from (6.27) and the relations for $f_\eta$ and $f_{\eta'}$ are given by (A.23) and (9.9). These are our main "predictions" for observable quantities. They are displayed in Table 6. We find a very satisfactory agreement with experiment for line $(b)$. The uncertainty in the "prediction" for $f_\eta$ and $f_{\eta'}$ is reflected in the differences as compared to lines $(a)$ and $(c)$. Also the leading mixing approximation $(d)$ is not too far from experiment, even though contributions beyond this approximation need to be included. The general tendency of the higher derivative contributions in the $\eta$–$\eta'$ sector is an enhancement of $f_{\eta'}$ and a decrease in $M_\eta$ and $f_\eta$ (cf. Fig. 8). From Table 7 we note that the octet decay constant $f_{\eta 8}$ is rather close to $f_K$ whereas the corresponding singlet decay constant almost equals $f_\pi$. In Table 7 we also show the mixing angles in the $\eta$–$\eta'$ sector. We find a large mixing between $\eta$ and $\eta'$ with an important dependence on the momentum. The mixing for $q^2 = -M_{\eta'}^2$ is substantially larger than that for $q^2 = -M_\eta^2$. The mixing corresponding to $\theta_p(\eta) \simeq -13.7^o$ is somewhat smaller than earlier estimates from chiral perturbation theory [2] where the size of the mixing angle was extracted indirectly from the requirement of a realistic value for $M_\eta$. On the other hand, the mixing corresponding to $\theta_p(\eta')$ is larger. We should point out that our direct method of computing all elements of the matrix for the inverse propagator in the $\eta$–$\eta'$ system is quite different from the indirect consistency requirement for $M_\eta$. We furthermore see in Table 7 a large deviation of $m_p^2/M_{\eta'}^2$ from one despite the fact that this difference is formally only a quadratic term in the quark mass expansion. We also present in Table 7 the isospin violation in the decay constants (6.24), $f_{K^0} - f_{K^\pm}$. It is reduced significantly as compared to the value obtained to lowest order in the quark mass expansion (2.12) or in chiral perturbation theory [2].

Our results in the scalar sector depend in addition on $Z_h/Z_m$, $\lambda_3\sigma_0^2$, $\overline{\omega}_h v$ and $\zeta_h v^2$. We use $\lambda_3\sigma_0^2$, $\overline{\omega}_h v$ and $\zeta_h v^2$ as input parameters together with a fixed value $M_{K_0^*} = 1430$ MeV. From (4.33), (4.34) we determine $m_h^2$ and $\gamma_6 v$ as functions of these three couplings. Here we use our "optimal values" for $\lambda_2$, $\nu$, etc. corresponding to the second line $(b)$ in Tables 2–7. We determine $Z_h/Z_m$ as a function of $\lambda_3$, $\overline{\omega}_h v$ and $\zeta_h v^2$ according to

**Table 7.** Mixing angles in the $\eta$–$\eta'$ system and isospin violation in the decay constants

|  | $\frac{f_{\eta 8}}{\text{MeV}}$ | $\frac{f_{\eta' 8}}{\text{MeV}}$ | $\frac{f_{\eta 0}}{\text{MeV}}$ | $\frac{f_{\eta' 0}}{\text{MeV}}$ | $\theta_p(\eta)$ | $\theta_p(\eta')$ | $\frac{m_p}{\text{MeV}}$ | $\frac{m_p^2}{M_{\eta'}^2}$ | $\frac{f_{K^0}-f_{K^\pm}}{\text{MeV}}$ |
|---|---|---|---|---|---|---|---|---|---|
| (a) | 116.6 | 132.1 | 95.3 | 107.9 | −15.6 | −31.2 | 839.1 | 0.77 | 0.25 |
| (b) | 117.1 | 124.2 | 96.3 | 102.1 | −13.7 | −28.0 | 865.9 | 0.82 | 0.28 |
| (c) | 116.0 | 126.6 | 94.0 | 102.6 | −16.3 | −31.1 | 839.8 | 0.77 | 0.22 |
| (d) | 117.7 | 120.4 | 97.5 | 99.8 | −14.7 | −28.6 | 843.5 | 0.78 | 0.30 |

**Table 8.** Masses and parameters of the scalar octet

| $\overline{\omega}_h v$ | $\zeta_h v^2$ | $\lambda_3 \sigma_0^2$ | $\frac{\hat{M}_{K_o^*}}{\text{MeV}}$ | $\frac{Z_h}{Z_m}$ | $\frac{m_h}{\text{MeV}}$ | $\frac{\gamma_6 v}{\text{MeV}^2}$ | $\frac{\gamma_2}{\text{MeV}}$ | $\frac{M_{a_o}}{\text{MeV}}$ | $\frac{M_{f_8}}{\text{MeV}}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.00 | 0.0 | 1430 | 0.35 | 1407 | $(254.7)^2$ | 8394 | 1360 | 1453 |
| −0.2 | 0.00 | 0.0 | 1430 | 0.39 | 1335 | $(241.6)^2$ | 7963 | 1178 | 1541 |
| 0.0 | 0.00 | 9.4 | 1430 | 0.35 | 1378 | $(382.6)^2$ | 7973 | 1267 | 1480 |
| −0.2 | 0.00 | 9.4 | 1430 | 0.39 | 1307 | $(363.0)^2$ | 7564 | 1097 | 1570 |
| 0.0 | 0.02 | 0.0 | 1295 | 0.42 | 1274 | $(230.6)^2$ | 7601 | 1232 | 1315 |
| 0.0 | 0.04 | 0.0 | 1144 | 0.54 | 1126 | $(203.7)^2$ | 6715 | 1088 | 1162 |

$$\frac{Z_h}{Z_m} = \tag{13.1}$$
$$\frac{m_g^2 + \sigma_0(3\sigma_0\lambda_2 + \nu) + \frac{1}{6}(9\sigma_0\lambda_2 - \nu + 12\sigma_0^3\lambda_3)[\overline{f}_K - \overline{f}_\pi]}{\left[1 + \frac{1}{2}\overline{\omega}_h v\right]\hat{M}_{K_o^*}^2}.$$

The results of this analysis and, in particular, the masses of the lowest lying $0^{++}$ octet are given for several values of $\lambda_3$, $\overline{\omega}_h v$ and $\zeta_h v^2$ in Table 8. Here $\lambda_3\sigma_0^2 = 9.4$ corresponds to the "maximal value" $\lambda_2/2$ compatible with a convergent expansion in $\sigma_0$ (cf. Sect. 10). From an estimate of $\zeta_h v^2$ due to the exchange of the vector field $\partial_\mu\rho_V^\mu$ (see Appendix B) we learn that the two last lines in Table 8 are preferred. Additional large mixing effects (see Sect. 11 and Appendix C) may further lower $M_{a_o}$ and lead to a mass of the isotriplet $a_0$ consistent with the observed resonance $a_0(980)$. The scalar partner of the $\eta$ appears to be associated with the broad resonance $f_0(1300)$. We finally emphasize the large deviation of $Z_h/Z_m$ from one which underlines the importance of nonminimal kinetic terms in the linear meson model.

## 14 Conclusions

In this paper we have investigated the effective action for the linear meson model. Including the discussion of the vector and axialvector fields from Appendix B a fairly simple picture emerges. Expressed in terms of scalar fields $\Phi$ and vector fields $\rho_V^\mu$, $\rho_A^\mu$ the quark mass expansion seems to converge rather well for the three light flavors. The same holds for the derivative expansion, leading to an approximate momentum dependence of propagators $\sim (Zq^2 + \overline{M}^2)^{-1}$. The only exception from this picture seems to be the scalar isotriplet $a_0(980)$ which can be explained (see Appendix C) by a large contribution of two–kaon or four–quark states.

The divergence $\partial_\mu\rho_A^\mu$ has the same quantum numbers as the pseudoscalar octet plus singlet. We have estimated the resulting mixing effect or, equivalently, the terms induced in the effective action for $\Phi$ from integrating out $\partial_\mu\rho_A^\mu$ ("partial Higgs effect"). We find a large non–minimal kinetic term which induces substantial quark mass corrections to the kinetic terms for the pseudoscalars ($\omega_m^{(\rho)}v \simeq -0.15$). This effect remains compatible with a converging quark mass expansion for the flavored pseudoscalars $\pi$ and $K$. On the other hand, an investigation of the masses $M_\eta$ and $M_{\eta'}$ as well as the decay constants $f_\eta$ and $f_{\eta'}$ shows that the non–minimal kinetic term induces in turn a large momentum dependent mixing in the $\eta$–$\eta'$ sector. Here we find that contributions which are of third or higher order in a formal quark mass expansion are comparable in size to the contributions arising to second order. It is the nonlinearity generated by this large mixing which leads to "predictions" for $M_\eta$, $f_\eta$ and $f_{\eta'}$ which are compatible with experimental observations. This explains why the measured value of $f_\eta$ is quite far away from its value for zero quark masses[20]. It is amazing to see that the nonlinearities in $M_\eta$, $f_\eta$ and $f_{\eta'}$ as functions of the variable $\omega_m v$ all conspire such that a common value of $\omega_m v$ can explain simultaneously these three quantities. Even more, our phenomenological estimate $\omega_m v \simeq -0.2$ is quite close to the estimate from the partial Higgs effect, $\omega_m^{(\rho)}v \simeq -0.15$. The latter involves completely different quantities like the $\rho\pi\pi$ coupling and the masses of the axialvectors!

It will be very interesting to see if our "phenomenological" picture of the effective linear meson model can be obtained from the solution of a flow equation similar to [6]. It would be highly nontrivial if the couplings would come out in such an approach in a range consistent with the analysis of the present paper. This concerns, in particular, the quartic coupling $\lambda_2$ which was found in [6] to be essentially determined by an infrared fixed point behavior. Furthermore, a more systematic analysis of the (axial)vector meson sector including quark mass effects should lead to a quantitative estimate of several effective cubic and quartic vertices relevant for the decay properties of these mesons. Beyond a successful explanation of the observed values for $f_\eta$ and $f_{\eta'}$ our results constitute the "phenomenological basis" for interesting further devel-

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[20] We emphasize once again that the nonlinearities in $\Delta$ or the poor convergence of an expansion in $\Delta$ appear only in the eigenvalues $M_\eta$ and $M_{\eta'}$, the mixing angles $\theta_p(\eta)$, $\theta_p(\eta')$ and the decay constants $f_\eta$, $f_{\eta'}$. The matrix elements of the $\eta$–$\eta'$ propagator converge satisfactorily

opments. They also shed light on the important question of the convergence of the quark mass expansion.

# Appendices

## A Meson decay constants

In this appendix we discuss the meson decay constants within the linear sigma model. Most results displayed here are well known from current algebra and are simply rephrased in a somewhat different language. The only slightly delicate issue concerns the choice of the normalization of fields. This determines the appropriate definition of wave function renormalization constants. A careful treatment of these constants is relevant for quantitative relations between decay constants and meson masses as discussed in the main text. We adopt here definitions of $f_\pi$, $f_K$, $f_\eta$, etc. which are directly related to measured partial decay widths of the corresponding mesons.

The weak leptonic decay of the charged pion, $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$, involves the effective three point vertex ($\bar{\gamma}$ being the Euclidean analog of $\gamma_5$)

$$
\begin{aligned}
\Gamma_{\pi\mu\bar{\nu}} = i \int \frac{d^4 p_\pi}{(2\pi)^4} \frac{d^4 p_\mu}{(2\pi)^4} \frac{d^4 p_{\bar{\nu}}}{(2\pi)^4} \\
\times \left[ \frac{g^2 \cos \vartheta_c}{4\sqrt{2} M_W^2} F_\pi^\rho(p_\pi, p_\mu, -p_{\bar{\nu}}) \right. \\
\times \pi^-(p_\pi)\bar{\mu}(p_\mu)\gamma_\rho(1+\bar{\gamma})\nu_\mu(-p_{\bar{\nu}})(2\pi)^4 \\
\left. \times \delta(p_\pi - p_\mu - p_{\bar{\nu}}) + \text{h.c.} \right] .
\end{aligned} \tag{A.1}
$$

Here we have projected onto the leptonic $V - A$ structure following from virtual $W$–exchange with $M_W$ the $W$–boson mass, $g$ the weak gauge coupling and $\vartheta_c$ the Cabibbo angle. (Analogously, the effective vertex for the charged kaon decay is obtained from (A.1) by the replacements $F_\pi^\rho \to F_K^\rho$, $\pi^- \to K^-$ and $\cos \vartheta_c \to \sin \vartheta_c$.) The vertex function $F_\pi^\rho$ can depend only on two independent momenta (e.g., $p_\pi$ and $p_{\bar{\nu}}$) and the leptonic pion decay involves its value for on–shell momenta, $p_\pi^2 = -M_{\pi^\pm}^2$, $p_\mu^2 = -M_\mu^2$, $p_{\bar{\nu}}^2 = 0$. In the present context we neglect the dependence of $F_\pi^\rho$ on the leptonic momenta and use the parameterization[21]

$$
F_\pi^\rho = p_\pi^\rho F_\pi(p_\pi^2) . \tag{A.2}
$$

Our task is therefore the evaluation of the pion decay constant

$$
f_\pi = F_\pi(p_\pi^2 = -M_{\pi^\pm}^2) \tag{A.3}
$$

which is determined experimentally from the leptonic decay width of the pion (up to electromagnetic corrections)

$$
\Gamma_{\pi \to \mu\bar{\nu}} = \frac{G_F^2}{4\pi} m_\mu^2 M_{\pi^\pm} f_\pi^2 \left(1 - \frac{m_\mu^2}{M_{\pi^\pm}^2}\right)^2 \cos^2 \vartheta_c \tag{A.4}
$$

---

[21] A term $\sim p_{\bar{\nu}}^\rho$ would not contribute to the pion decay anyhow, since on shell we have $\not{p}_{\bar{\nu}}\nu(-p_{\bar{\nu}}) = 0$

derived from (A.1) with $G_F = \sqrt{2}g^2/(8M_W^2)$.

In order to compute $F_\pi$ we consider the linear $\sigma$–model coupled to external currents. For this purpose we replace all derivatives acting on $\Phi$ by covariant ones

$$
\begin{aligned}
D^\mu \Phi &= \partial^\mu \Phi - \frac{i}{2}\lambda_z R_z^\mu \Phi + \frac{i}{2}\Phi\lambda_z L_z^\mu \\
&= \partial^\mu \Phi - \frac{i}{2}V_z^\mu [\lambda_z, \Phi] - \frac{i}{2}A_z^\mu \{\lambda_z, \Phi\} .
\end{aligned} \tag{A.5}
$$

Here the vector– and axialvector currents $V_z^\mu$ and $A_z^\mu$ are related to the left and right handed currents $L_z^\mu$ and $R_z^\mu$, respectively, by

$$
\begin{aligned}
L_z^\mu &= V_z^\mu - A_z^\mu \\
R_z^\mu &= V_z^\mu + A_z^\mu .
\end{aligned} \tag{A.6}
$$

By this replacement $\Gamma_{\text{kin}} = \int d^4 x \mathcal{L}_{\text{kin}}$ (4.6) becomes a functional of the (local) background fields $V^\mu$ and $A^\mu$. Current conservation is automatically embodied in this construction. The coupling of mesons to $W$–bosons can be infered via the identification

$$
\begin{aligned}
L_{1,2}^\mu &= g \cos \vartheta_c W_{1,2}^\mu \\
L_{4,5}^\mu &= g \sin \vartheta_c W_{1,2}^\mu .
\end{aligned} \tag{A.7}
$$

Once the couplings of mesons to $W$–bosons are known the couplings to lepton pairs follow by insertion of the field equation

$$
\begin{aligned}
W_i^\mu(p) = -\frac{g}{4} \frac{1}{p^2 + M_W^2} \\
\times \int \frac{d^4 q}{(2\pi)^4} \bar{\psi}(q-p)\tau_i \gamma^\mu(1+\bar{\gamma})\psi(q) . \tag{A.8}
\end{aligned}
$$

Here $\psi$ stands for the lepton doublets and we will neglect $p^2$ as compared to $M_W^2$. Extracting the coefficient linear in $L_{1,2}^\mu$

$$
\begin{aligned}
\Gamma_L &= -\int \frac{d^4 p}{(2\pi)^4} K_{L,z}^\mu(p) L_{z,\mu}(p) \\
&= -\frac{i}{2\sqrt{2}} \int \frac{d^4 p}{(2\pi)^4} \pi^-(p)F_\pi^\mu(p) \\
&\quad \times [L_{1,\mu}(p) - iL_{2,\mu}(p)] + \dots \tag{A.9}
\end{aligned}
$$

we can relate $F_\pi$ to the part of $K_{L,1}^\mu$ which is linear in $\pi_1$

$$
K_{L,1}^\mu(p) = \frac{i}{2}p^\mu F_\pi(p)\pi_1(p) + \dots . \tag{A.10}
$$

The discussion of leptonic decays of charged kaons is analogous with

$$
K_{L,4}^\mu(p) = \frac{i}{2}p^\mu F_K(p)K_4(p) + \dots . \tag{A.11}
$$

Here $\pi_{1,2}$ and $K_{4,5}$ are the pion and kaon fields, $\pi^- = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)$, $K^- = \frac{1}{\sqrt{2}}(K_4 + iK_5)$, with standard normalization of their kinetic terms such that their inverse propagator (two point function) in the vicinity of its zero at timelike momenta is given by $p^2 + M_{\pi^\pm}^2 (M_{K^\pm}^2)$.

For an evaluation of the terms linear in the currents $V_z^\mu$, $A_z^\mu$ and linear in the meson fields one covariant derivative should be replaced by

$$D^\mu \Phi \longrightarrow -\frac{i}{2} V_z^\mu [\lambda_z, \langle \Phi \rangle] - \frac{i}{2} A_z^\mu \{\lambda_z, \langle \Phi \rangle)\} \qquad (A.12)$$

with $\langle \Phi \rangle = \mathrm{diag}(\varphi_u, \varphi_d, \varphi_s)$ the expectation value of $\Phi$. All other covariant derivatives have to act as simple derivatives on the meson fields. Consider first the limit of vanishing quark masses where $\varphi_u = \varphi_d = \varphi_s = \overline{\sigma}_0$. The term linear in the vector current $V_z^\mu$ vanishes whereas the axial current couples linearly to the pion field as a consequence of spontaneous chiral symmetry breaking, i.e. $D_\mu \Phi \to -i\overline{\sigma}_0 A_z^\mu \lambda_z$. The relevant term in $\mathcal{L}_{\mathrm{kin}}$ (4.6) reads

$$\mathcal{L}_{\mathrm{kin}} \longrightarrow -\frac{1}{2} \left( Z_\varphi + X_\varphi^- \overline{\sigma}_0^2 + U_\varphi \overline{\sigma}_0 \right)$$
$$\times Z_m^{-1} \sigma_0 A_z^\mu \, \mathrm{Tr}\, \{\lambda_z, \partial_\mu m\} \qquad (A.13)$$

and, with $Z_m = Z_\varphi + X_\varphi^- \overline{\sigma}_0^2 + U_\varphi \overline{\sigma}_0$ (cf. Sect. 8), one infers

$$K_{L,z}^\mu = -\frac{1}{4} \sigma_0 \, \mathrm{Tr}\, \{\lambda_z, \partial_\mu m\} \ . \qquad (A.14)$$

In the limit of vanishing quark masses the fields $m_z$ are already properly normalized and we can identify $\pi_1 = m_1$, $K_1 = m_4$. This yields

$$f_\pi = 2\sigma_0 \ . \qquad (A.15)$$

The proportionality between $f_\pi$ and $\sigma_0$ is no surprise. It is well known that in the limit of vanishing quark masses the pion decay is related to the non–conservation of the axial current induced by chiral symmetry breaking.

Going beyond the limit of vanishing quark masses the expectation values of $\varphi_u$, $\varphi_d$ and $\varphi_s$ are different. Nevertheless, there is no term linear both in the vector current $V_z^\mu$ and a pseudoscalar meson. This follows from the discrete symmetries $\mathcal{C}$, $\mathcal{P}$ and is related to the observation that the first term on the right hand side of (A.12) is hermitean whereas the second is anti–hermitean. On the other hand, the vector current couples linearly to the scalars as, for example (see (4.4), (4.11))

$$\mathrm{Tr}\, (D^\mu \Phi)^\dagger D_\mu \Phi \qquad (A.16)$$
$$\longrightarrow -\frac{1}{2} A_z^\mu \, \mathrm{Tr}\, \left[ \{\lambda_z, \langle \Phi \rangle\} \left( Z_m^{-\frac{1}{2}} \partial_\mu m + \frac{2}{\sqrt{6}} Z_p^{-\frac{1}{2}} \partial_\mu p \right) \right]$$
$$-\frac{i}{2} V_z^\mu \, \mathrm{Tr}\, \left[ [\lambda_z, \langle \Phi \rangle] \left( Z_h^{-\frac{1}{2}} \partial_\mu h + \frac{2}{\sqrt{6}} Z_s^{-\frac{1}{2}} \partial_\mu s \right) \right] \ .$$

For our computation of $f_\pi$ we can omit the term linear in $V_z^\mu$.

Let us consider first the approximation where the kinetic term (4.6) is truncated to $Z_\varphi \, \mathrm{Tr}\, (D^\mu \Phi)^\dagger (D_\mu \Phi)$. In this limit $m$ describes already the properly normalized pion and kaon fields. One finds

$$K_{L,1}^\mu = \frac{i}{2} p^\mu \pi_1(p) Z_\varphi Z_m^{-\frac{1}{2}} (\varphi_u + \varphi_d) \qquad (A.17)$$

or

$$f_\pi = Z_\varphi Z_m^{-\frac{1}{2}} (\varphi_u + \varphi_d) \ . \qquad (A.18)$$

Similarly, we find for the leptonic decay of the charged kaons

$$f_{K^\pm} = Z_\varphi Z_m^{-\frac{1}{2}} (\varphi_u + \varphi_s) \qquad (A.19)$$

and define

$$f_{K^0} = Z_\varphi Z_m^{-\frac{1}{2}} (\varphi_d + \varphi_s) \ . \qquad (A.20)$$

Noting $\overline{\sigma}_0 = \frac{1}{3} (\varphi_u + \varphi_d + \varphi_s)$ and observing that in this approximation $Z_m = Z_\varphi$ one arrives at (5.6)

$$f_\pi + f_{K^\pm} + f_{K^0} = 6\overline{\sigma}_0 Z_m^{\frac{1}{2}} = 6\sigma_0 \ . \qquad (A.21)$$

With the definitions $\Delta_u = Z_m^{\frac{1}{2}} \varphi_u - \sigma_0$ etc. and $\langle h \rangle = \sqrt{2} Z_h^{\frac{1}{2}} \langle \Phi_s \rangle = 2 Z_h^{\frac{1}{2}} (\langle \Phi \rangle - \overline{\sigma}_0) = 2 (Z_h/Z_m)^{\frac{1}{2}} \mathrm{diag}(\Delta_u, \Delta_d, \Delta_s)$ we obtain the relations (2.6).

Next we consider the more general kinetic term (4.6). The first effect is a nontrivial wave function renormalization between the fields $m_1$ and $\pi_1$, i.e.

$$m_1 = Z_\pi^{-\frac{1}{2}} \pi_1 \ , \quad m_4 = Z_{K^\pm}^{-\frac{1}{2}} K_4 \ . \qquad (A.22)$$

This effect multiplies $f_\pi$ by a factor $Z_\pi^{-\frac{1}{2}}$ and similarly for $f_{K^\pm}$, $f_{K^0}$. Here we note that $Z_\pi$, $Z_{K^\pm}$ and $Z_{K^0}$ should be defined at the corresponding poles such that the inverse renormalized two–point function is approximated in the vicinity of the pole by $q^2 + M_{\pi^\pm}^2$ with $M_{\pi^\pm} \simeq 135.1$ MeV the physical pion mass (after subtraction of electromagnetic effects). The second effect reflects the modification of the general kinetic term into which (A.5) is inserted. Since the axialvector field $A_\mu(q)$ is needed for on–shell momenta, we conclude that all kinetic terms must be evaluated at the poles. Expanding the inverse propagators around $q^2 = -M^2$, knowledge of the coefficient linear in $q^2$ suffices for a computation of the meson decay constants. For the pions this is given by $Z_\pi$ and the evaluation of the full kinetic term gives a factor $Z_\pi$ in the formula for $f_\pi$. This can be seen directly by inserting (A.5) into the contributions $\sim X_\varphi^-$, $U_\varphi$ (8.6), (8.9) and using the relations (8.10). In summary, the total effect of the generalized kinetic term is a multiplication of $f_\pi$ with $Z_\pi^{1/2}$. Similarly, $f_{K^\pm}$ and $f_{K^0}$ are proportional to $Z_{K^\pm}^{1/2}$ and $Z_{K^0}^{1/2}$, respectively. This explains the relations (6.9). We emphasize that $Z_\pi$, $Z_{K^\pm}$ and $Z_{K^0}$ should be evaluated from the coefficient linear in $q^2$ in an expansion of the inverse propagator around $q^2 = -M_{\pi^\pm}^2, -M_{K^\pm}^2, -M_{K^0}^2$, respectively. More precisely, they are defined by (6.11) with $q_0^2$ replaced by $-M_i^2$.

We finally extend the discussion to the decay constants of the non–flavored pseudoscalars $f_{\pi^0}$, $f_\eta$ and $f_{\eta'}$. By (9.9) and (9.10) they are related to the couplings of these mesons to the corresponding components of the axialvector currents, or in a different language, the expectation value of the corresponding current between the vacuum and the meson state. For instance, $f_{\eta 8}$ parameterizes the coupling of $\eta$ to the current $A_8$ whereas $f_{\eta' 0}$ corresponds to the coupling of the $\eta'$ to the singlet current.

For a comparison of $f_{\eta 8}$ with $f_\pi$ we therefore have to replace $\mathrm{Tr}\left(\{\lambda_1, \langle\Phi\rangle\}\lambda_1\right) Z_\pi^{1/2}$ by $\mathrm{Tr}\left(\{\lambda_8, \langle\Phi\rangle\}\lambda_8\right) Z_8^{1/2}$. The ratios of $f_{\pi^0}$, $f_{\eta 8}$ and $f_{\eta'0}$ to $f_\pi$ are then easily computed

$$\frac{f_{\pi^0}}{f_\pi} = \left(\frac{Z_{\pi^0}}{Z_\pi}\right)^{\frac{1}{2}} \tag{A.23}$$

$$\frac{f_{\eta 8}}{f_\pi} = \left(\frac{Z_8}{Z_\pi}\right)^{\frac{1}{2}} \frac{\varphi_u + \varphi_d + 4\varphi_s}{3(\varphi_u + \varphi_d)}$$

$$= \left(\frac{Z_8}{Z_\pi}\right)^{\frac{1}{2}} \frac{2\overline{f}_{K^\pm} + 2\overline{f}_{K^0} - \overline{f}_\pi}{3\overline{f}_\pi} \tag{A.24}$$

$$\frac{f_{\eta 0}}{f_\pi} = \left(\frac{Z_8}{Z_\pi}\right)^{\frac{1}{2}} \frac{2(\varphi_u + \varphi_d + \varphi_s)}{3(\varphi_u + \varphi_d)}$$

$$= \left(\frac{Z_8}{Z_\pi}\right)^{\frac{1}{2}} \frac{\overline{f}_{K^\pm} + \overline{f}_{K^0} + \overline{f}_\pi}{3\overline{f}_\pi}. \tag{A.25}$$

Similarly, the octet and singlet decay constants for the normalization of the $\eta'$, $f_{\eta'8}$ and $f_{\eta'0}$, are given by

$$\frac{f_{\eta'8}}{f_{\eta 8}} = \frac{f_{\eta'0}}{f_{\eta 0}} = \left(\frac{Z_p}{Z_m Z_8}\right)^{\frac{1}{2}} \tilde{z}_p(-M_{\eta'}^2)^{1/2}. \tag{A.26}$$

Here $\tilde{z}_p(-M_{\eta'}^2)$ accounts for higher derivative effects and is normalized for $q^2 = -M_\eta^2$ according to $\tilde{z}_p(-M_\eta^2) = 1$. If the higher derivative effects are omitted one also has $\tilde{z}_p(-M_{\eta'}^2) = 1$. Furthermore, if we neglect the mixing effects (or for $z_8(q^2) = z_p(q^2)$) we can identify $\tilde{z}_p(q^2)$ with $z_p(q^2)$ appearing in (6.26). We note that (A.26) is appropriate for a definition of $Z_p$ at $q_0^2 = -M_\eta^2$. If one instead would define $Z_p$ at $q_0^2 = -M_{\eta'}^2$ the factor $z_p(-M_{\eta'}^2)^{1/2}$ would be absorbed by this alternative definition.

# B Vector mesons

In this section we briefly discuss[22] the vector and pseudovector fields and their interactions with scalars and pseudoscalars. This will permit us to estimate the part of the effective interactions involving four (pseudo)scalars which is induced by the exchange of vector fields. We introduce the fields $\rho_L^\mu$ and $\rho_R^\mu$ as hermitian $3 \times 3$ matrices which transform as $\mathbf{8} \oplus \mathbf{1}$ under $SU_L(3)$ and $SU_R(3)$, respectively, being neutral with respect to the other part of the flavor group. The interaction with (constituent) quarks

$$\mathcal{L}_{\rho\overline{q}q} = \frac{1}{\sqrt{2}} Z_q g_{\rho\overline{q}q} \left(\overline{q}_L \gamma_\mu \rho_L^\mu q_L + \overline{q}_R \gamma_\mu \rho_R^\mu q_R\right) \tag{B.1}$$

respects $SU_L(3) \times SU_R(3)$ and is also consistent with left–right symmetry ($\rho_L \leftrightarrow \rho_R$) and charge conjugation ($\rho_L \to -\rho_R^T$, $\rho_R \to -\rho_L^T$). The invariant kinetic and mass terms read

$$\mathcal{L}_{\rho,2} = \frac{Z_\rho}{8} \mathrm{Tr}\left(\partial_\mu \rho_{L\nu} - \partial_\nu \rho_{L\mu}\right)\left(\partial^\mu \rho_L^\nu - \partial^\nu \rho_L^\mu\right)$$

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[22] See also [21] and references therein

$$+ \frac{Z_\rho}{4\alpha_\rho} \mathrm{Tr}\left(\partial_\mu \tilde{\rho}_L^\mu\right)^2 + \frac{Z_\rho}{12\alpha'_\rho}\left(\partial_\mu \mathrm{Tr}\,\rho_L^\mu\right)^2$$

$$+ \frac{1}{4} m_\rho^2 \mathrm{Tr}\,\tilde{\rho}_{L\mu}\tilde{\rho}_L^\mu + \frac{1}{12} m_\rho'^2 \mathrm{Tr}\,\rho_L^\mu \mathrm{Tr}\,\rho_{L\mu}$$

$$+ (L \to R) \tag{B.2}$$

where

$$\tilde{\rho}_{L,R}^\mu = \rho_{L,R}^\mu - \frac{1}{3} \mathrm{Tr}\,\rho_{L,R}^\mu \equiv \tilde{\rho}_{L,R}^{z,\mu}\lambda_z \tag{B.3}$$

denotes the octets and $\frac{1}{\sqrt{6}} \mathrm{Tr}\,\rho_{L,R}^\mu$ represents the singlets. In general, the field $\rho^\mu$ can describe spin–one and spin–zero ($\sim \partial_\mu \rho^\mu$) particles. For $\alpha_\rho, \alpha'_\rho \to 0$ the spin–zero components decouple and the fields $\rho^\mu$ only describes spin–one bosons. ($\partial_\mu \rho^\mu = 0$). In the opposite limit $\alpha_\rho, \alpha'_\rho \to \infty$ there remains a kinetic term only for the spin–one bosons whereas $\partial_\mu \rho^\mu$ can be determined algebraically from the field equations.

There is only one possible cubic coupling involving two scalars or pseudoscalars and the vector octet. To lowest order in a derivative expansion it reads

$$\mathcal{L}_{\Phi^2\rho} = \frac{i}{\sqrt{2}} g_{\rho\pi\pi} \hat{Z} \mathrm{Tr}\left[\left(\partial_\mu \Phi^\dagger \Phi - \Phi^\dagger \partial_\mu \Phi\right) \tilde{\rho}_L^\mu \right.$$
$$\left. + \left(\partial_\mu \Phi \Phi^\dagger - \Phi \partial_\mu \Phi^\dagger\right) \tilde{\rho}_R^\mu\right]. \tag{B.4}$$

The appropriate value of $\hat{Z}$ will be determined later such that $g_{\rho\pi\pi}$ appears in the width of the decay $\rho \to \pi\pi$ according to

$$\Gamma(\rho \to \pi\pi) = \frac{g_{\rho\pi\pi}^2}{48\pi} \frac{\left(M_\rho^2 - 4M_\pi^2\right)^{\frac{3}{2}}}{M_\rho^2}. \tag{B.5}$$

Using the experimental values $\Gamma(\rho \to \pi\pi) \simeq 150$ MeV and $M_\rho \simeq 770$ MeV this yields

$$g_{\rho\pi\pi} \simeq 6.0. \tag{B.6}$$

A second coupling appears for the singlets

$$\mathcal{L}'_{\Phi^2\rho} = \frac{i}{3} \frac{g'_{\rho\pi\pi}}{\sqrt{2}} \hat{Z}$$
$$\times \mathrm{Tr}\left(\partial_\mu \Phi^\dagger \Phi - \Phi^\dagger \partial_\mu \Phi\right)\left(\mathrm{Tr}\,\rho_L^\mu - \mathrm{Tr}\,\rho_R^\mu\right). \tag{B.7}$$

We note that similarly the couplings of the quarks to the vector and pseudovector singlets could be different from the octet couplings leading to a modification of (B.1). In the following we will neglect for simplicity the differences between the singlets and octets (i.e., $g'_{\rho\pi\pi} = g_{\rho\pi\pi}$, $m_\rho'^2 = m_\rho^2$, $\alpha'_\rho = \alpha_\rho$). Accordingly, we give for the quartic couplings involving two (pseudo)scalars and two (pseudo)vectors only those appearing for the octets and extend them to the singlets. In contrast to the cubic coupling (B.4) the lowest order term does not involve derivatives:

$$\mathcal{L}_{\Phi^2\rho^2} = \frac{1}{2} \hat{Z}\left(g_{\rho\pi\pi}^2 + f_1\right)\left(\mathrm{Tr}\,\Phi^\dagger \Phi \rho_L^\mu \rho_{L\mu} + \mathrm{Tr}\,\Phi\Phi^\dagger \rho_R^\mu \rho_{R\mu}\right)$$
$$- \hat{Z}\left(g_{\rho\pi\pi}^2 + f_2\right) \mathrm{Tr}\,\Phi^\dagger \rho_R^\mu \Phi \rho_{L\mu}$$
$$+ \hat{Z} f_3 \left(\mathrm{Tr}\,\Phi^\dagger \Phi - \rho_0\right) \mathrm{Tr}\left(\rho_L^\mu \rho_{L\mu} + \rho_R^\mu \rho_{R\mu}\right). \tag{B.8}$$

Our conventions are such that for $f_1 = f_2 = f_3 = 0$ the couplings (B.4), (B.7), (B.8) and an appropriate scalar kinetic term can be written in terms of a gauge covariant derivative

$$D_\mu \Phi = \partial_\mu \Phi - \frac{i}{\sqrt{2}} g_{\rho\pi\pi} \rho_{R\mu} \Phi + \frac{i}{\sqrt{2}} g_{\rho\pi\pi} \Phi \rho_{L\mu} \quad \text{(B.9)}$$

as $\hat{Z} \operatorname{Tr} (D^\mu \Phi)^\dagger (D_\mu \Phi)$. For $g_{\rho\bar{q}q} = g_{\rho\pi\pi}$ this also extends to the couplings to quarks.

Chiral symmetry breaking by the expectation value of $\Phi$ leads to a mixing between $\rho_L^\mu$ and $\rho_R^\mu$. In the absence of quark masses the $\rho$–mass matrix reads

$$\overline{M}_{VA}^2 = \quad \text{(B.10)}$$
$$\begin{pmatrix} m_\rho^2 + 2 \left(g_{\rho\pi\pi}^2 + f_1\right) \hat{Z}\overline{\sigma}_0^2, & -2 \left(g_{\rho\pi\pi}^2 + f_2\right) \hat{Z}\overline{\sigma}_0^2 \\ -2 \left(g_{\rho\pi\pi}^2 + f_2\right) \hat{Z}\overline{\sigma}_0^2, & m_\rho^2 + 2 \left(g_{\rho\pi\pi}^2 + f_1\right) \hat{Z}\overline{\sigma}_0^2 \end{pmatrix} .$$

The mass eigenstates are the vector and pseudovector mesons (and associated scalars)

$$\begin{aligned} \rho_V^\mu &= \tfrac{1}{\sqrt{2}} \left(\rho_R^\mu + \rho_L^\mu\right) \\ \rho_A^\mu &= \tfrac{1}{\sqrt{2}} \left(\rho_R^\mu - \rho_L^\mu\right) . \end{aligned} \quad \text{(B.11)}$$

They transform under charge conjugation as

$$\begin{aligned} \rho_V &\xrightarrow{C} -\rho_V^T \\ \rho_A &\xrightarrow{C} \rho_A^T \end{aligned} \quad \text{(B.12)}$$

and we conclude that the transversal parts of $\rho_V^\mu$ and $\rho_A^\mu$ describe the $1^{--}$ and $1^{++}$ octets and singlets of the light meson spectrum. The mass of the vector mesons is given by

$$M_V^2 = m_\rho^2 + 2 \left(f_1 - f_2\right) \hat{Z}\overline{\sigma}_0^2 \quad \text{(B.13)}$$

whereas the pseudovector mass reads

$$\overline{M}_A^2 = m_\rho^2 + 2 \left(2g_{\rho\pi\pi}^2 + f_1 + f_2\right) \hat{Z}\overline{\sigma}_0^2 . \quad \text{(B.14)}$$

We have put here a bar on $\overline{M}_A^2$ in order to indicate that the relation with physical axialvector masses involves an additional wave function renormalization

$$M_A^2 = \frac{\overline{M}_A^2}{Z_A} . \quad \text{(B.15)}$$

In fact, chiral symmetry breaking induces also a difference in the kinetic terms for $\rho_V$ and $\rho_A$ by invariants of the type $\operatorname{Tr} \Phi^\dagger F_R^{\mu\nu} \Phi F_{L\mu\nu}$ with $F_{L,R}^{\mu\nu} = \partial^\mu \rho_{L,R}^\nu - \partial^\nu \rho_{L,R}^\mu$. Combining the most general term involving up to two powers of $\Phi$

$$\Delta\mathcal{L}_{\text{kin},\rho} = \frac{\beta_1}{4} \operatorname{Tr} \Phi^\dagger F_R^{\mu\nu} \Phi F_{L\mu\nu}$$
$$+ \frac{\beta_2}{8} \operatorname{Tr} \left(\Phi^\dagger \Phi F_L^{\mu\nu} F_{L\mu\nu} + \Phi\Phi^\dagger F_R^{\mu\nu} F_{R\mu\nu}\right) \quad \text{(B.16)}$$

with (B.2) one finds for the kinetic term relevant for the spin–one bosons in the limit of vanishing quark masses

$$\mathcal{L}_{\text{kin},\rho} = \frac{Z_V}{8} \operatorname{Tr} F_V^{\mu\nu} F_{V\mu\nu} + \frac{Z_A}{8} \operatorname{Tr} F_A^{\mu\nu} F_{A\mu\nu} \quad \text{(B.17)}$$

with

$$\begin{aligned} Z_V &= Z_\rho + (\beta_2 + \beta_1)\overline{\sigma}_0^2 \\ Z_A &= Z_\rho + (\beta_2 - \beta_1)\overline{\sigma}_0^2 . \end{aligned} \quad \text{(B.18)}$$

The wave function renormalization $Z_\rho$ can be fixed by convention. If we choose the normalization of $\rho_{L,R}^\mu$ such that the kinetic term for $\rho_V^\mu$ has the standard form ($Z_V = 1$) there remains nevertheless an additional parameter $Z_A$ multiplying the kinetic term of $\rho_A^\mu$ which typically differs from one. Similar effects can influence the effective kinetic terms for the spin–zero components $\sim \partial_\mu \rho^\mu$.

We observe that for $g_{\rho\pi\pi}^2 + f_2 > 0$ the vector octet is indeed lighter than the pseudovector octet. For $|f_2| \ll g_{\rho\pi\pi}^2$ the mass splitting can be related to the $\rho\pi\pi$ coupling and therefore to the $\rho$ lifetime by

$$\overline{M}_A^2 - M_V^2 = 4 \left(g_{\rho\pi\pi}^2 + f_2\right) \hat{Z}\overline{\sigma}_0^2$$
$$= \frac{1}{9} \left(2\overline{f}_K + \overline{f}_\pi\right)^2 g_{\rho\pi\pi}^2 \frac{\hat{Z}}{Z_m} x_\rho \quad \text{(B.19)}$$

with

$$x_\rho = 1 + \frac{f_2}{g_{\rho\pi\pi}^2} . \quad \text{(B.20)}$$

The mass splitting within the octet because of nonvanishing quark masses can also be understood from the interactions (B.8) and (B.16) inserting $\Phi = \overline{\sigma}_0 + \frac{1}{2}(w\lambda_3 - \sqrt{3}v\lambda_8)Z_h^{-1/2}$. (This can be used to determine the parameters appearing in these expressions.)

It is instructive to write the cubic coupling (B.4) in terms of mass eigenstates by using (4.4):

$$\begin{aligned} \mathcal{L}_{\Phi^2\rho} = \frac{i}{2} g_{\rho\pi\pi} \hat{Z} \operatorname{Tr} \Bigg\{ & \left[\partial_\mu \Phi_p \Phi_p - \Phi_p \partial_\mu \Phi_p \right. \\ & + \partial_\mu \Phi_s \Phi_s - \Phi_s \partial_\mu \Phi_s ] \rho_V^\mu + i\left[\partial_\mu \Phi_p \Phi_s - \Phi_p \partial_\mu \Phi_s \right. \\ & + \Phi_s \partial_\mu \Phi_p - \partial_\mu \Phi_s \Phi_p + \frac{2}{3} \left(\chi_s \partial_\mu \chi_p - \partial_\mu \chi_s \chi_p\right) \\ & + \frac{2}{\sqrt{3}} \left(\chi_s \partial_\mu \Phi_p - \partial_\mu \chi_s \Phi_p \right. \\ & \left. \left. - \chi_p \partial_\mu \Phi_s + \partial_\mu \chi_p \Phi_s\right) \right] \rho_A^\mu \Bigg\} \\ & + \sqrt{2} g_{\rho\pi\pi} \hat{Z}\overline{\sigma}_0 \operatorname{Tr} \left\{ \left(\Phi_p + \frac{1}{\sqrt{3}} \chi_p\right) \partial_\mu \rho_A^\mu \right\}. \quad \text{(B.21)} \end{aligned}$$

We note a mixing of the longitudinal component $\partial_\mu \rho_A^\mu$ with the pseudoscalar mesons $\Phi_p$, $\chi_p$ for $\overline{\sigma}_0 > 0$. This is possible since $\partial_\mu \rho_A^\mu$ represents a $0^{-+}$ state. In contrast, $\partial_\mu \rho_V^\mu$ transforms as $0^{+-}$ and a mixing with $0^{++}$ states $\Phi_s$ is possible only for the charged scalars. It is induced by a nonvanishing expectation value $\langle \Phi_s \rangle$. As required by $\mathcal{C}$ and $\mathcal{P}$ invariance the vector mesons have cubic couplings to two pseudoscalars only if those are distinct, e.g. there is a $\rho^0 \pi^+ \pi^-$ but no $\rho^0 \pi^0 \pi^0$ coupling. Typical decays of pseudovectors involve the coupling of $\rho_A$ to one pseudoscalar and one scalar. Additional cubic couplings involving two

(pseudo)vectors and one (pseudo)scalar are generated by (B.8) if $\overline{\sigma}_0$ is inserted for one of the fields $\Phi$.

For an estimate of the effective $\Phi$–interactions induced by the exchange of $\rho$–fields we have to solve the field equations for $\rho_V^\mu$ and $\rho_A^\mu$ as functions of $\Phi$. The result is then inserted into the action thus eliminating $\rho^\mu$ and replacing it by functions of $\Phi$. We will do so keeping only terms linear and quadratic in $\Phi$. This is sufficient for invariants containing up to four powers of $\Phi$. For the contributions from the quartic term (B.8) we keep only the lowest order $\Phi = \text{diag}(\overline{\sigma}_0)$, i.e. we neglect the mass splitting within the octets. The resulting field equations for $\rho_V^\mu$ and $\rho_A^\mu$ are

$$\left( -\partial^\nu \partial_\nu \delta_\mu^\sigma + \left( 1 - \frac{1}{\alpha_\rho} \right) \partial^\sigma \partial_\mu + M_V^2 \delta_\mu^\sigma \right) \rho_{V,\sigma}^{ab}$$

$$= -i g_{\rho\pi\pi} \hat{Z} \Big( \partial_\mu \Phi^\dagger \Phi - \Phi^\dagger \partial_\mu \Phi$$

$$+ \partial_\mu \Phi \Phi^\dagger - \Phi \partial_\mu \Phi^\dagger \Big)^{ab}$$

$$\left( -\partial^\nu \partial_\nu \delta_\mu^\sigma + \left( 1 - \frac{1}{\alpha_\rho} \right) \partial^\sigma \partial_\mu + \overline{M}_A^2 \delta_\mu^\sigma \right) \rho_{A,\sigma}^{ab}$$

$$= i g_{\rho\pi\pi} \hat{Z} \Big( \partial_\mu \Phi^\dagger \Phi - \Phi^\dagger \partial_\mu \Phi$$

$$- \partial_\mu \Phi \Phi^\dagger + \Phi \partial_\mu \Phi^\dagger \Big)^{ab} \quad \text{(B.22)}$$

If we omit first effects from chiral symmetry breaking one has $M_V^2 = \overline{M}_A^2 = m_\rho^2$. Inserting (B.22) into (B.2) and (B.4) and keeping only terms involving up to two derivatives, the vector and pseudovector mesons contribute to the kinetic term (4.6) only a structure

$$X_\varphi^{-(\rho)} = -\frac{4 g_{\rho\pi\pi}^2 \hat{Z}^2}{m_\rho^2} \; . \quad \text{(B.23)}$$

For $\overline{\sigma}_0 > 0$ the $SU_L(3) \times SU_R(3)$ symmetry is spontaneously broken. The effective interactions mediated by $\rho_V$ and $\rho_A$ still preserve the vector–like $SU(3)$ symmetry if quark mass effects are neglected. In this approximation it is most convenient to give directly the contribution from $\rho$–exchange to the wave function renormalization constants $Z_m$, $Z_p$, $Z_h$ and $Z_s$ as well as $\omega_m$, $\omega_{pm}$ and $\omega_h$. They can be read off from (B.21)

$$Z_m^{(\rho)} = Z_p^{(\rho)} = -\frac{4 g_{\rho\pi\pi}^2}{\overline{M}_A^2} \hat{Z}^2 \overline{\sigma}_0^2$$

$$Z_h^{(\rho)} = Z_s^{(\rho)} = \omega_h^{(\rho)} = 0$$

$$\omega_m^{(\rho)} = -\frac{4 g_{\rho\pi\pi}^2}{\overline{M}_A^2} \frac{\hat{Z}^2}{Z_h^{\frac{1}{2}} Z_m} \overline{\sigma}_0 \quad \text{(B.24)}$$

$$\omega_{pm}^{(\rho)} = -\frac{8}{\sqrt{6}} \frac{g_{\rho\pi\pi}^2}{\overline{M}_A^2} \frac{\hat{Z}^2}{(Z_h Z_m Z_p)^{\frac{1}{2}}} \overline{\sigma}_0 \; .$$

This yields the same expressions as inserting (B.23) into (8.10) if $m_\rho^2$ is replaced by $\overline{M}_A^2$. Indeed, to linear order in the quark masses there is no contribution to the wave function renormalization constants from the exchange of the vector field $\rho_V^\mu$. Only the exchange of the spin–zero component $\partial_\mu \rho_A^\mu$ induces the corrections (B.24). This can

easily be understood from the structure of the interactions (B.21). Possible contributions to the quadratic terms defining $Z_m$, $Z_h$, $\omega_m$ etc. can only arise through terms in the field equations for $\rho$ which are linear in $\Phi - \langle \Phi \rangle$. By Lorentz–invariance such terms must be $\sim \int d^4 x \partial_\mu \Phi \rho^\mu \sim \int d^4 x (\Phi - \langle \Phi \rangle) \partial_\mu \rho^\mu$. The discrete symmetries $\mathcal{C}$ and $\mathcal{P}$ allow to this order only a term $\sim \Phi_p \partial_\mu \rho_A^\mu$ which corresponds to the mixing of the $0^{-+}$ states and a structure $\sim [\Phi_s - \langle \Phi_s \rangle, \langle \Phi_s \rangle] \partial_\mu \rho_V^\mu$ for the mixing in the scalar sector. The mixing in the pseudoscalar sector receives contributions $\sim \sigma_0, v$ and therefore contributes to the terms in (B.24). In contrast, the mixing in the scalar sector vanishes for $v = 0$ and gives therefore only a correction (11.2) to the scalar kinetic terms which is quadratic in $v$

$$\zeta_h^{(\rho)} = \frac{1}{4} \frac{\hat{Z}^2}{Z_h^2} \frac{g_{\rho\pi\pi}^2}{M_V^2} \; . \quad \text{(B.25)}$$

The contribution to $Z_m^{(\rho)}$ (B.24) is known as the "partial Higgs effect". Inserting (B.14) for $\overline{M}_A^2$ one finds for $f_1 = f_2 = 0$ and $\hat{Z} = Z_\varphi$

$$Z_m = Z_\varphi + Z_m^{(\rho)} = Z_\varphi \frac{m_\rho^2}{\overline{M}_A^2} \quad \text{(B.26)}$$

and observes that $Z_m$ vanishes for $m_\rho^2 \to 0$. In this limit the symmetry $SU_L(3) \times SU_R(3)$ becomes an exact local gauge symmetry. As a result of the Higgs effect the pseudoscalar octet disappears from the spectrum. What remains are massive pseudovector mesons which acquire their mass through spontaneous symmetry breaking of the axial $SU_A(3)$ by $\overline{\sigma}_0 \neq 0$. In the real world, however, the local gauge symmetry is explicitly broken by the mass term $m_\rho^2$ and by the deviation of $f_1$ and $f_2$ from zero. Below we will take $\hat{Z}$ different from $Z_\varphi$ and this again violates local $SU_L(3) \times SU_R(3)$ symmetry and modifies (B.26). The partial Higgs effect corresponds to the mixing between the two $0^{-+}$ octet states contained in $\Phi_p$ and $\partial_\mu \rho_A^\mu$. We may further improve the estimate (B.24) by taking into account the contributions $\sim q^2$ on the left hand side of (B.22). The propagator of the divergence of $\rho_A^\mu$ differs from the one for the pseudovector mesons, since the kinetic term is given by the "gauge fixing" term $\sim \frac{1}{\alpha_\rho}$ in (B.2). (There are similar differences in higher order in the momentum $q^2$ and in contributions to the kinetic terms from chiral symmetry breaking.) The expression for $\overline{M}_A^2$ appropriate in (B.24) should involve the inverse propagator for the longitudinal component of $\rho_A^\mu$, $Z_P q^2 + \overline{M}_A^2$, where to lowest order $Z_P = \alpha_\rho^{-1}$. It should be taken at a momentum scale corresponding to the light pseudoscalar octet masses, $q_0^2 = -m_m^2$. We also define a renormalized mass parameter

$$M_P^2 = \overline{M}_A^2 / Z_P \; . \quad \text{(B.27)}$$

Inserting this into (B.24) yields the relation

$$\left( \frac{Z_h}{Z_m} \right)^{\frac{1}{2}} \omega_m^{(\rho)} \left( 2 \overline{f}_K + \overline{f}_\pi \right)$$

$$= -\frac{2}{3}\frac{g_{\rho\pi\pi}^2}{\overline{M}_A^2 - Z_P m_m^2}\left(2\overline{f}_K + \overline{f}_\pi\right)^2 \frac{\hat{Z}^2}{Z_m^2}$$

$$= -\frac{6}{x_\rho}\frac{\hat{Z}}{Z_m}\frac{M_A^2 - M_V^2/Z_A}{M_A^2 - m_m^2 Z_P/Z_A} \qquad (B.28)$$

where we used (B.19) for the last equation. We note that unless $Z_P/Z_A = M_A^2/M_P^2$ is very large the precise value of $Z_P$ has only little influence. In fact, we will see that $Z_P/Z_A \lesssim 0.35$ (cf. Table 9). The average masses of the vectors and pseudovectors are given by ($M_\rho = 770$ MeV, $M_{K^*} = 892$ MeV, $M_{a_1} = 1230$ MeV, $M_{K_1} = 1340$ MeV)

$$M_V^2 = \tfrac{2}{3}M_{K^*}^2 + \tfrac{1}{3}M_\rho^2 \simeq (853\text{ MeV})^2$$
$$M_A^2 = \tfrac{2}{3}M_{K_1}^2 + \tfrac{1}{3}M_{a_1}^2 \simeq (1300\text{ MeV})^2 . \qquad (B.29)$$

This leads to a quantitative estimate for the $\rho$–exchange contribution to $\omega_m v$

$$\omega_m^{(\rho)}v = -\frac{4}{x_\rho}\frac{\hat{Z}}{Z_m}\frac{(\overline{f}_K - \overline{f}_\pi)}{(2\overline{f}_K + \overline{f}_\pi)}\frac{(1 - M_V^2/M_A^2)}{(1 - m_m^2/M_P^2)}$$
$$\times \left[1 - \frac{M_V^2}{M_A^2 - M_V^2}\left(\frac{1}{Z_A} - 1\right)\right]$$
$$\simeq -\frac{0.28}{x_\rho}\frac{\hat{Z}}{Z_m}\left[1 - 0.75\left(\frac{1}{Z_A} - 1\right)\right] \qquad (B.30)$$

where in the last step we have neglected the weak dependence on $M_P^2$. At this stage we see already that a typical order of magnitude for $\omega_m^{(\rho)}v$ is around $-0.2$ which is quite close to what is needed to explain $f_\eta$ and $f_{\eta'}$ (see Sect. 9).

For a more detailed estimate we have to determine the appropriate choice of $\hat{Z}$. After elimination of $\rho_A^\mu$ by solving its fields equation the relevant terms in the effective interaction for $\Phi$ and $\rho_V$ are

$$\mathcal{L}_{\Phi V} = Z_m \text{Tr}(\partial^\mu \Phi)^\dagger \partial_\mu \Phi + \frac{i}{2}g_{\rho\pi\pi}\hat{Z}$$
$$\times \text{Tr}\left[\left(\partial_\mu\Phi^\dagger\Phi - \Phi^\dagger\partial_\mu\Phi + \partial_\mu\Phi\Phi^\dagger - \Phi\partial_\mu\Phi^\dagger\right)\rho_V^\mu\right] . \qquad (B.31)$$

As a result of the partial Higgs effect we notice (B.24) the appearance of

$$Z_m = Z_\varphi - 4g_{\rho\pi\pi}^2\frac{\overline{\sigma}_0^2}{\overline{M}_A^2 - Z_P m_m^2}\hat{Z}^2 \qquad (B.32)$$

instead of[23] $Z_\varphi$ in front of the kinetic term for $\Phi$. Within the lowest order derivative approximation employed here we therefore have to identify

$$\hat{Z} = Z_m \qquad (B.33)$$

in order to get the standard coupling of $\rho_V$ to the renormalized scalar field. Indeed, the lowest order $\rho\pi\pi$ interaction now takes the form

$$\mathcal{L}_{\rho\pi\pi} = g_{\rho\pi\pi}\epsilon_{ijk}\pi_j\partial_\mu\pi_k\rho_i^\mu$$
$$= -2ig_{\rho\pi\pi}\rho_3^\mu\pi^+\partial_\mu\pi^- + \dots \qquad (B.34)$$

---

[23] More precisely, $Z_\varphi$ stands in this appendix for $Z_\varphi + U_\varphi\overline{\sigma}_0 + \tilde{X}_\varphi^-\overline{\sigma}_0^2$ where $\tilde{X}_\varphi^-$ is the part of $X_\varphi^-$ which is unrelated to "$\rho$–exchange"

**Table 9.** The table gives the "leading mixing" results for various parameters related to $\eta$–$\eta'$ mixing. Only in the last line we use a fixed value $Z_p/Z_m = 0.9$ corresponding to a nonvanishing $U_\varphi$ in (8.10)

| $x_\rho$ | $\frac{Z_p}{Z_m}$ | $Z_A$ | $\frac{\overline{M}_A}{\text{MeV}}$ | $Z_P$ | $\frac{M_P}{\text{MeV}}$ | $\frac{M_0}{\text{MeV}}$ | $\omega_m^{(\rho)}v$ | $Z_m/Z_\varphi$ | $\delta_\omega^{(\rho)}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.00 | 0.67 | 1068 | 0.00 | – | – | −0.14 | 0.73 | 0.00 |
| 0.8 | 0.99 | 0.62 | 1030 | 0.22 | 2196 | 1852 | −0.17 | 0.71 | 0.08 |
| 1.0 | 0.99 | 0.67 | 1069 | 0.22 | 2278 | 1944 | −0.15 | 0.73 | 0.07 |
| 1.2 | 0.99 | 0.72 | 1106 | 0.22 | 2358 | 2031 | −0.14 | 0.74 | 0.07 |
| 1.0 | 0.9 | 0.67 | 1069 | 0.16 | 2673 | 2282 | −0.15 | 0.73 | 0.05 |

as can be seen by inserting $\rho_V^\mu = \rho_i^\mu\tau_i$, $\Phi = \frac{1}{2}Z_m^{-1/2}f_\pi \exp\left(i\frac{\pi_k\tau_k}{f_\pi}\right)$, $i,j,k = 1,2,3$. With the choice (B.33) the interactions (B.4), (B.7) can be combined for $f_1 = f_2 = 0$ into a covariant kinetic term for $\Phi$

$$\mathcal{L}'_{\Phi V} = Z_m \text{Tr}\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)$$
$$D_\mu\Phi = \partial_\mu\Phi - \frac{i}{2}g_{\rho\pi\pi}\left[\rho_{V\mu},\Phi\right] . \qquad (B.35)$$

We observe in passing that because of the violation of gauge symmetry by the mass term $\sim m_\rho^2\rho_A^\mu\rho_{A\mu}$ one cannot have simultaneously local gauge invariance of the interaction terms with respect to $SU_L(3) \times SU_R(3)$ (corresponding to $\hat{Z} = Z_\varphi$) and the vector–like subgroup $SU_V(3)$ after spontaneous symmetry breaking (corresponding to $\hat{Z} = Z_m$). We could, of course, replace in (B.4) $g_{\rho\pi\pi}Z_m = \hat{g}_{\rho\pi\pi}Z_\varphi$. The quartic interactions (B.8), however, would then be proportional to $Z_m g_{\rho\pi\pi}^2 = Z_\varphi\hat{g}_{\rho\pi\pi}^2\left(1 + \frac{Z_\varphi - Z_m}{Z_m}\right)$. In this normalization (B.35) corresponds to nonvanishing $\hat{f}_1 = \hat{f}_2 = \frac{Z_\varphi - Z_m}{Z_m}$ and a different normalization of the gauge coupling. We will consider here the case of approximate $SU_V(3)$ gauge symmetry (B.35) in contrast to part of the literature where the choice $\hat{Z} = Z_\varphi$ is adopted while $\hat{f}_1$ and $\hat{f}_2$ are neglected. In our case, the interaction terms are only invariant with respect to global chiral $SU_L(3) \times SU_R(3)$ transformations. Local $SU_V(3)$ invariance arises effectively only after chiral symmetry breaking if the symmetry breaking terms $m_\rho^2$, $f_i$, etc. are neglected.

The wave function renormalization $Z_A$ can be evaluated from the average mass of the pseudovector mesons (B.19)

$$Z_A = \frac{M_V^2 + 4g_{\rho\pi\pi}^2\sigma_0^2 x_\rho}{M_A^2} . \qquad (B.36)$$

If we neglect the weak dependence of $\omega_m^{(\rho)}v$ on $Z_P$, (B.30), (B.33) and (B.36) are already sufficient to compute $\omega_m^{(\rho)}v$ as a function of $x_\rho$. For a more accurate estimate we have to determine $Z_P$ or $M_P$ which will be done in Appendix C (cf. (C.5)). This allows to determine $Z_A$, $\overline{M}_A$, $Z_P$, $M_P$ and $\omega_m^{(\rho)}v$ for given $x_\rho$. The results are displayed in Table 9 for different values of $x_\rho$ and $M_0$ (cf. Appendix C). The first five lines are evaluated in the leading mixing approximation as described in the end of Sect. 12. The last

line assumes in addition $U_\varphi \neq 0$ in (8.10) and corresponds to line $(d)$ in Sect. 13. The main uncertainties in $\omega_m^{(\rho)} v$ arise from $x_\rho \neq 1$ and from possible higher derivative contributions to the decay $\rho \to \pi\pi$ which would modify the value of $g_{\rho\pi\pi}$. In addition, there are also corrections from the mass splitting in the vector– and pseudovector octets. The latter are suppressed by an additional power of quark masses. We believe that the values for $Z_m/Z_\varphi$ in Table 9 give a rather realistic estimate for the quantitative importance of the "partial Higgs effect". For the values in Table 9 we have assumed $\omega_m = \omega_m^{(\rho)}$ for the computation of $\overline{f}_K - \overline{f}_\pi$, $\sigma_0$, etc. If we use instead the values of line $(b)$ in Table 4, $\omega_m^{(\rho)} v$ typically decreases by 0.02 thus reaching values close to $-0.2$. In conclusion, typical values for $\omega_m^{(\rho)} v$ are in the range between $-0.14$ and $-0.17$. This coincides more or less with the values needed for an explanation of the observed decay constants $f_\eta$ and $f_{\eta'}$! We also note that the partial Higgs effect reflected in the deviation of $Z_m/Z_\varphi$ from one leads typically to a 30% correction in the kinetic terms.

Let us finally turn to the size of the mixing effects in the scalar sector parameterized by $\zeta_h$ (B.25). Despite their suppression by a factor $\sim v^2$ they may be quantitatively relevant because the propagator of the exchanged quantum now involves the mass of the vector instead of the axialvector mesons. Since this mixing concerns only the $K_0^*$ mesons (cf. (11.2)) we use for a quantitative estimate of $\zeta_h$ the mass $M_{K^*} = 892$ MeV and replace in (B.25) $M_V^2$ by $M_{K^*}^2 + \tilde{Z}_P q_0^2 = M_{K^*}^2 - \tilde{Z}_P M_{K_o^*}^2$. (Here $\tilde{Z}_P$ differs from $Z_P$ by contribution $\sim \overline{\sigma}_0^2$ similar to (B.16)). This implies

$$
\begin{aligned}
\zeta_h^{(\rho)} v^2 &= \frac{1}{9} \frac{Z_m}{Z_h} \left( \frac{g_{\rho\pi\pi}(\overline{f}_K - \overline{f}_\pi)}{M_{K^*}} \right)^2 \left( 1 - \tilde{Z}_P \frac{M_{K_o^*}^2}{M_{K^*}^2} \right)^{-1} \\
&\simeq \frac{0.0185}{1 - 2.57 \tilde{Z}_P}
\end{aligned}
\tag{B.37}
$$

and we note that this formula is not applicable for $\tilde{Z}_P$ around 0.4. There $M_{K^*}^2/\tilde{Z}_P$ is of the same size as $M_{K_o^*}^2$ and the mixing of states cannot be described by a derivative expansion anymore (cf. Appendix C). Also the imaginary part of the propagators of unstable particles have to be taken into account. Because of the uncertainty in the value of $\tilde{Z}_P$ it is difficult to give a precise quantitative estimate of $\zeta_h^{(\rho)} v^2$. We only know that this quantity is positive and exceeds the value for $Z_P = 0$. This leads to the estimate $\zeta_h v^2 \gtrsim 0.02$.

We conclude that the "classical" exchange of spin one mesons only contributes to the effective quartic (and higher) interactions of the pseudoscalars but does not modify their effective propagators. They are therefore not relevant for the investigations of the present work which concentrate on masses and mixings. The only contributions from higher states to the propagators concern the mixing with higher $0^{-+}$ states for $\Phi_p$ and $\chi_p$ and additional scalar states for $\Phi_s$. Such states are contained in $\partial_\mu \rho_A^\mu$ and $\partial_\mu \rho_V^\mu$. These mixings contribute to quantities like $X_\varphi^-$, $Z_h/Z_m$ etc. A brief general discussion of mixing effects is given in Appendix C.

## C Mixing with other states

In QCD the pseudoscalar and scalar mesons described by the field $\Phi$ are only part of a rich spectrum of quark–antiquark states plus glueballs and possibly also $\overline{q}q\overline{q}q$ states. There are strong couplings between the various states and their physics therefore influences the behavior of the $0^{-+}$ and $0^{++}$ particles described in this work. On the level of effective propagators which are the main subject of this paper we have to consider mixing effects with other $0^{-+}$ or $0^{++}$ states in the spectrum. For the pseudoscalar octet this concerns only the higher mass $0^{-+}$ octets, whereas for the scalar octet we also have to consider the possible mixing with $\overline{q}q\overline{q}q$ states in the $0^{++}$ channel. We mention that it is not crucial in this context if the two–meson or four–quark states (with $\overline{q}q\overline{q}q$ quantum numbers) correspond to "particles" like the $a_0(980)$, $f_0(980)$ or not — the composite fields describing the $\overline{q}q\overline{q}q$ states may also have "propagators" without a pole. Finally, for the pseudoscalar $\eta'$ we also have to include a mixing with pseudoscalar glueballs. We collectively denote these additional resonances as "higher states".

The general method for dealing with the higher states is to integrate them out and to compute an effective theory for $\Phi$ alone. The investigations of this paper should be understood in this context. There are various methods for integrating out the higher states. One consists in computing first the effective action including additional fields for the higher states. In a second step the field equations for these states are solved for arbitrary values of $\Phi$. The resulting "classical fields" are functionals of $\Phi$ and can be reinserted into the effective action, thus leading to an effective action which depends only on $\Phi$. The discussion in Appendix B can serve as an example. For those results of the present paper which are only based on symmetries it is actually not necessary to perform the integration of additional fields in practice. Nevertheless, some insight in the origin of some of the constants of the effective action, like $X_\varphi^-$, $\tilde{V}_\varphi$ etc., can be gained by considering the possible form of the effective action including additional states. We should point out that we neglect throughout the imaginary part of the two–point functions which is due to the decay of unstable resonances. This approximation may become invalid in the immediate vicinity of poles in the propagators.

We have already encountered the mixing of $0^{-+}$ states in the discussion of the longitudinal component of $\rho_A^\mu$ in Appendix B. Let us rephrase this with a somewhat different perspective by introducing a field

$$
\tau_P = \partial_\nu \rho_A^\nu - \frac{1}{3} \text{Tr} \, \partial_\nu \rho_A^\nu
\tag{C.1}
$$

for the additional $0^{-+}$ state. With this normalization the propagator for $\tau_P$ can be approximated by $q^2 G_P(q^2)$ with

$G_P^{-1} = Z_P q^2 + \overline{M}_A^2$ (cf. Appendix B). The inverse propagator for the coupled system of $\sqrt{2}\Phi_p$ and $\tau_P$ contains off–diagonal terms

$$G^{-1}(q) = \begin{pmatrix} G_\varphi^{-1}(q) \,, \; G_{\varphi P}^{-1}(q) \\ G_{P\varphi}^{-1}(q) \,, \; G_P^{-1}(q)/q^2 \end{pmatrix} \qquad (C.2)$$

which are responsible for the mixing. From (B.21) one finds

$$G_{\varphi P}^{-1}(q) = G_{P\varphi}^{-1}(q) = 2g_{\rho\pi\pi} Z_m \overline{\sigma}_0 \,. \qquad (C.3)$$

A similar mixing occurs between $\chi_p$ and the singlet $\tau_P'$ contained in $\mathrm{Tr}\,\partial_\mu \rho_A^\mu$. It is no accident that the mixing vanishes for $\overline{\sigma}_0 = 0$ or $q^2 = 0$: In the limit of unbroken chiral symmetry ($\overline{\sigma}_0 = 0$) the fields $\Phi$ and $\rho$ belong to different representations of $SU_L(3) \times SU_R(3)$ and cannot mix. Also for $\overline{\sigma}_0 \neq 0$ and vanishing quark masses $\Phi_p$ describes Goldstone bosons which can only have derivative couplings. By construction the quark mass terms only appear as source terms for $\Phi$.

It is equivalent to diagonalize the matrix (C.2) or to eliminate $\tau_P$ by solving its field equations for $\tau_P[\Phi]$ which is more adapted to our purpose. The elimination of $\tau_P$ gives an additional contribution to the effective inverse propagator $G_\varphi^{-1}(q) + \Delta G_\varphi^{-1}(q)$, namely

$$\Delta G_\varphi^{-1}(q) = -G_{\varphi P}^{-2}(q) G_P(q) q^2 \,. \qquad (C.4)$$

There are also contributions to the off–diagonal $m - p$ kinetic term related to $\eta$–$\eta'$ mixing which are represented graphically in Fig. 9. From an investigation of $G_\varphi^{-1} + \Delta G_\varphi^{-1}$ we can obtain some general insight in the structure of mixing effects. First, we observe that for $G_\varphi^{-1} = Z_\varphi q^2 + \overline{M}^2$ the effective propagator $\overline{G} = (G_\varphi^{-1} + \Delta G_\varphi^{-1})^{-1}$ typically has two poles corresponding to the values of $q^2$ for which the determinant of $G^{-1}(q)$ vanishes. In the vicinity of the lower mass pole $G_\varphi^{-1} + \Delta G_\varphi^{-1}$ can be approximated by a typical one particle two point function. The propagator vanishes at the value $q^2 = -M_P^2$ where $G_P^{-1}(q)$ has a zero. One observes that due to the particular factor of $(q^2)^{-1}$ in the inverse $\tau_P$ propagator (C.2) the value of $M_P$ is always larger than both masses corresponding to the location of the poles[24]. In addition, the residue of $\overline{G}$ at the pole with the higher value of $-q^2$ has the opposite sign as for the lower mass pole. (The higher mass pole does not correspond to a stable particle even within our approximations.) If we denote by $q^2 = -M_0^2$ the location of the higher pole one finds with (C.3) and $\overline{M}^2 = m_m^2 Z_m$

$$M_P^2 = \overline{M}_A^2/Z_P \qquad (C.5)$$
$$= M_0^2 \left[ 1 + 4g_{\rho\pi\pi}^2 \sigma_0^2 \frac{1}{Z_P} \frac{Z_m}{Z_\varphi} \left( M_0^2 - m_m^2 \frac{Z_m}{Z_\varphi} \right)^{-1} \right] \,.$$

An estimate of $M_0$ is not obvious and subject to large uncertainties. One can then determine $Z_\varphi/Z_m$ from (B.32),

---
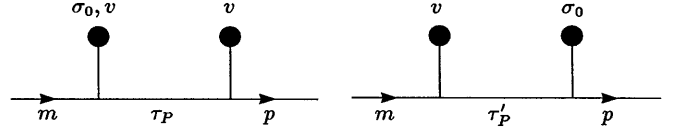[24] Without the $(q^2)^{-1}$ factor $M_P^2$ would be inbetween the two poles



**Fig. 9.** Feynman diagrams contributing to $\eta$–$\eta'$ mixing due to exchange of $\partial_\mu \rho_A^\mu$

(B.33) and solve the resulting system of equations in dependence on $Z_P$. Results are displayed in Table 9 for values of $Z_P$ in a range for which $M_\eta$ comes out with a reasonable size in the leading mixing approximation (Fig. 8). We note that much larger values of $Z_P$ lead to a very large mixing in the $\eta$–$\eta'$ sector and completely destroy any reasonable picture. We also show in Table 9 three different values of $x_\rho$.

Second, it is clear that for real octet or singlet fields the off–diagonal elements must be real and equal. Integrating out the additional fields gives a negative contribution to the coefficient $\sim q^2$ in the quadratic term for $\Phi - \langle\Phi\rangle$ as long as $G_P(q)$ stays positive. The mixing gives therefore a negative contribution to $Z_m$ (8.10) and explains why $X_\varphi^-$ is negative. Third, from (C.4) we learn that the mixing effects are proportional to the propagator $G_P$. This suggests that mixing effects with light additional states are particularly important. Fourth, the mixing effects also contribute to higher derivative terms in the effective action for $\Phi$. Expanding $G_P(q)$ around $q_0^2$ gives

$$\Delta G_\varphi^{-1} = -4 \frac{g_{\rho\pi\pi}^2 Z_m^2 \overline{\sigma}_0^2}{\overline{M}_A^2 + Z_P q_0^2} q^2 \left[ 1 - \frac{Z_P (q^2 - q_0^2)}{\overline{M}_A^2 + Z_P q_0^2} \right] \qquad (C.6)$$

where the first term corresponds to $Z_m^{(\rho)} q^2$ as given by (B.24). Comparing with (6.12) and using (B.24) we find a contribution

$$\overline{H}_m^{(\rho)} = -\frac{1}{4} \omega_m^{(\rho)} v \frac{2\overline{f}_K + \overline{f}_\pi}{\overline{f}_K - \overline{f}_\pi} \frac{1}{M_P^2 - m_m^2} \,. \qquad (C.7)$$

The momentum dependence from the effective propagator $G_P$ may be particularly important for the $\eta'$ since its mass is closest to $M_P^2$. The resulting $q^2$–dependence of the quantities appearing in (6.26) has therefore to be treated with care. We remark that typical masses in the neutral sector are higher than in the flavored one and therefore guess $M_P$ around 2000 MeV with a large uncertainty. In terms of the unrenormalized fields the inverse propagator in the flavor neutral sector takes on the form

$$\begin{pmatrix} \left[ Z_\varphi + X_\varphi^-(q^2) \overline{\sigma}_0^2 \right] q^2 + M_p^2 Z_p, & \left[ \hat{\omega}(q^2) q^2 + M_{8p}^2 \right] Z_m^{1/2} Z_p^{1/2} \\ \left[ \hat{\omega}(q^2) q^2 + M_{8p}^2 \right] Z_m^{1/2} Z_p^{1/2}, & \left[ Z_\varphi + X_\varphi^-(q^2) \overline{\sigma}_0^2 \right] q^2 + \overline{M}_8^2 \end{pmatrix} . \qquad (C.8)$$

Here we assume that the nontrivial momentum dependence beyond the approximation linear in $q^2$ arises dominantly from propagator effects contained in $X_\varphi^-(q^2)$ and $\hat{\omega}(q^2)$ according to

$$X_\varphi^-(q^2) = X_\varphi^-(-m_m^2) \frac{(M_P^2 - m_m^2)}{(M_P^2 + q^2)} \qquad (C.9)$$

$$\hat{\omega}(q^2) = \hat{\omega}(-M_\eta^2) f_\omega(q^2) , \quad f_\omega(q^2) = \frac{M_P^2 - M_\eta^2}{M_P^2 + q^2} .$$

Our conventions imply

$$\hat{\omega} = \hat{\omega}(-M_\eta^2)$$
$$X_\varphi^- = X_\varphi^-(-m_m^2) \qquad \text{(C.10)}$$

and we remind that we have defined both, $Z_p$ and $Z_8$, for $q^2 = -M_\eta^2$. The leading mixing approximation implies

$$Z_m = Z_\varphi + X_\varphi^-(-m_m^2)\bar{\sigma}_0^2$$
$$Z_m Z_8 = Z_p = Z_\varphi + X_\varphi^-(-M_\eta^2)\bar{\sigma}_0^2 . \qquad \text{(C.11)}$$

With this definition the only higher derivative effect in the matrix (6.26) for $q^2 = -M_\eta^2$ appears in the factor $f_\omega^{-1}(-m_m^2) = (M_P^2 - m_m^2)/(M_P^2 - M_\eta^2) = 1.02$, (6.33). Here we have used the estimate $M_P = 2670$ MeV from Table 9. Also the resulting deviation of $Z_p/Z_m$ from unity is small

$$\begin{aligned}
\frac{Z_p}{Z_m} = Z_8 &= \frac{Z_\varphi + X_\varphi^-(-M_\eta^2)\bar{\sigma}_0^2}{Z_\varphi + X_\varphi^-(-m_m^2)\bar{\sigma}_0^2} \\
&= 1 + \left[ X_\varphi^-(-M_\eta^2) - X_\varphi^- \right] \sigma_0^2 \\
&= 1 + \frac{1}{4}\omega_m v \frac{(2\bar{f}_K + \bar{f}_\pi)}{(\bar{f}_K - \bar{f}_\pi)} \frac{(M_\eta^2 - m_m^2)}{(M_P^2 - M_\eta^2)} \\
&\simeq 1 + 0.05\omega_m v \qquad \text{(C.12)}
\end{aligned}$$

and compatible with the linearization for $Z_8$ according to (C.7). (The numerical value for $\omega_m v = -0.20$ is $Z_p/Z_m = 0.99$.) There is no modification of the decay constants $f_{\eta 8}$ and $f_{\eta 0}$, (A.24), (A.25). On the other hand, the propagator effect in $X_\varphi^-(q^2)$ and $\hat{\omega}(q^2)$ could lead to substantial effects for $q^2 = -M_{\eta'}^2$ depending on the value of $M_P$: In the diagonal elements of the inverse propagator (6.26) one has to insert

$$\begin{aligned}
z_p(q^2) = z_8(q^2) &= \frac{Z_\varphi + X_\varphi^-(q^2)\bar{\sigma}_0^2}{Z_\varphi + X_\varphi^-(-M_\eta^2)\bar{\sigma}_0^2} \\
&= \frac{\left(1 - \frac{1}{4}\omega_m v \frac{(2\bar{f}_K+\bar{f}_\pi)}{(\bar{f}_K-\bar{f}_\pi)} \frac{(q^2+m_m^2)}{(q^2+M_P^2)}\right)}{\left(1 + \frac{1}{4}\omega_m v \frac{(2\bar{f}_K+\bar{f}_\pi)}{(\bar{f}_K-\bar{f}_\pi)} \frac{(M_\eta^2-m_m^2)}{(M_P^2-M_\eta^2)}\right)} \qquad \text{(C.13)}
\end{aligned}$$

whereas $\hat{\omega}$ is replaced by $\hat{\omega}(q^2)$ (6.33), (C.9). Furthermore, the correct definition of the decay constants $f_{\eta'0}$ and $f_{\eta'8}$ involves now the factor (A.26)

$$\tilde{z}_p(-M_{\eta'}^2) = z_p(-M_{\eta'}^2) . \qquad \text{(C.14)}$$

For the scalar octet an interesting possibility of mixing concerns states in the two–meson channels. In fact, the four–point function for $\Phi_p$ may develop resonance–like structures in the momentum range corresponding to the sum of two pseudoscalar meson masses. (These momentum dependent structures are not accounted for by the four–point function at zero external momenta described by the effective potential $U$.) Such resonance structures

can be replaced by effective interactions with a composite $0^{++}$ field $\tau_S$. The effective two–point functions for $\tau_S$ obtained in this way do not necessarily correspond to a propagating particle or resonance, since their real part may be strictly positive and bounded for all values of $q^2$ on the real axis. Consider a generic structure for the mixing between $\sqrt{2}\Phi_s$ and $\tau_S$

$$G^{-1}(q) = \begin{pmatrix} Zq^2 + \overline{M}^2 , & b(q^2) \\ b(q^2) , & c(q^2) \end{pmatrix} . \qquad \text{(C.15)}$$

If $c(q^2)$ has a zero for $\sqrt{-q^2}$ in the vicinity of the sum of two pseudoscalar masses one finds two values of $q^2$ for which an eigenvalue of $G^{-1}$ vanishes. In the case of the isospin triplet they could be associated with $a_0(980)$ and $a_0(1320)$. On the other hand, the two–particle threshold could also be reflected by a finite enhancement of $c(q)^{-1}$ or $b(q)$ without a zero of $c(q)$. If $c(q^2)$ dips in this momentum region to values smaller than $b^2(q^2)/(\overline{M}^2 + Zq^2)$ the zero eigenvalue of $G^{-1}$ will occur precisely in the threshold region, namely for $q^2 = -M_0^2$ as determined by $c(-M_0^2) = b^2(-M_0^2)(\overline{M}^2 - ZM_0^2)^{-1}$. After solving for $\tau_S[\Phi_s]$ the location of the single pole of $[G_\varphi^{-1}(q) - b^2(q^2)c^{-1}(q^2)]^{-1}$ would then necessarily be found at $-M_0^2$ in the threshold region. For $\overline{M}^2/Z$ not too far from the two–particle threshold this effect could explain naturally why the isotriplet in $\Phi_s$ is found precisely at the $2K$ threshold! Mixing effects from $\Delta G_\varphi^{-1} = -b^2(q^2)c^{-1}(q^2)$ are large in this case. Since this mechanism requires a critical strength for $b^2 c^{-1}$ not all members of the scalar octet have to be in the vicinity of two–particle thresholds. More precisely, the phenomenon of "threshold mass shifting" which induces mesons masses near a two–particle threshold occurs whenever $\overline{M}^2/Z$ is above the threshold and the quantity $b^2(q)/(c(q)Z(q))$ makes a strong enough jump in the threshold region. An alternative way of looking at this "threshold mass shifting" notes that the loop contribution to the two–point function for the $a_0$ becomes important if the mass is close to the sum of the masses of the two pseudoscalars circulating in the loop.

Even without a detailed discussion of the complicated analyticity properties we conclude that for both alternatives the effective inverse propagator for the isotriplet in $\Phi_s$ should have a zero at the observed $a_0(980)$ resonance. In this momentum region the mixing effects with two–kaon states are expected to be very strong. No detailed understanding of the properties of the $a_0(980)$ seems possible without incorporating the two–kaon channel. In addition, the effective inverse propagator may (or may not) have a second zero corresponding to the possible resonance $a_0(1320)$. In this momentum region the mixing effects should be much smaller because of the larger value of $Zq^2 + \overline{M}^2 - c(q)$. We note that both resonances are described by the same value of[25] $\overline{M}^2$, but different effective $Z_h$ and $\omega_h$. The two different associations $a_0(980)$

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[25] It is convenient to choose composite fields such that $c(0) = 0$

vs. $a_0(1320)$ in the main text become in this case only two facets of the same story. If the $a_0(1320)$ exists the values characterizing the potential, like $m_h^2$, $\lambda_2$, $\lambda_3$, etc. should be independent of the identification of the isotriplet. The actual differences in these values are then a measure for the influence of neglected terms. We find that these differences can indeed be small if the mixing is large enough for the $a_0(980)$. This is compatible with the existence of the $a_0(1320)$ as a real resonance.

Finally, we turn to the mixing of a pseudoscalar $0^{-+}$ glueball $g$ with the $\eta'$. This is particularly interesting in view of a possible experimental detection of $g$. Because of the anomaly the $\eta'$ is not a Goldstone boson for vanishing quark masses and the off–diagonal element in the mixing matrix may not vanish for zero momentum. We introduce for the glueball a pseudoscalar singlet field $g$ with an effective action

$$\mathcal{L}_\mathrm{g} = \frac{1}{2}\partial^\mu g\partial_\mu g + \frac{1}{2}m_{gl}^2 g^2 + h_{gl}\omega g$$
$$\omega = i\left(\det\Phi - \det\Phi^\dagger\right) . \tag{C.16}$$

The coupling between $\Phi$ and $g$ conserves all symmetries $(\mathcal{P}(g) = -g, \mathcal{C}(g) = g)$. The real coupling $h_{gl}$ may depend on the momentum of $g$. Expanding $h_{gl}(q)$ around $q_0^2 = -m_m^2$ the mixing with the glueball contributes to the effective potential for $\Phi$ (cf. 4.8)

$$U^{(g)}[\Phi] = -\frac{1}{2}h_{gl}^2(q_0)\left[m_{gl}^2 + q_0^2\right]^{-1}\omega^2 \tag{C.17}$$

whereas for the kinetic term (4.6) it induces a coupling

$$\tilde{V}_\varphi^{(g)} = h_{gl}^2(q_0)\left[m_{gl}^2 + q_0^2\right]^{-2} - \frac{\partial h_{gl}^2}{\partial q^2}(q_0)\left[m_{gl}^2 + q_0^2\right]^{-1} . \tag{C.18}$$

It is probably difficult to disentangle (C.17) from other contributions to the potential. On the other hand, a determination of the size of the parameter $\tilde{V}_\varphi$ will put restrictions on $h_{gl}$. Since $h_{gl}$ is directly related to the mixing between $\eta'$ and $g$ one may obtain from it interesting information on the decay of the pseudoscalar glueball into mesons or photons. To lowest order we can use

$$\omega = -\sqrt{6}\bar{\sigma}_0^2 Z_p^{-\frac{1}{2}}p \tag{C.19}$$

and obtain the inverse propagator for the $\eta'$–glueball system as

$$G^{-1}(q) \simeq \tag{C.20}$$
$$\begin{pmatrix} q^2 + m_p^2, & -\frac{\sqrt{6}}{2}Z_p^{-\frac{1}{2}}Z_m^{-1}\sigma_0^2 h_{gl} \\ -\frac{\sqrt{6}}{2}Z_p^{-\frac{1}{2}}Z_m^{-1}\sigma_0^2 h_{gl}, & q^2 + m_{gl}^2 \end{pmatrix} .$$

For not too large $h_{gl}$ the mixing angle between $g$ and $\eta'$ is suppressed by the small ratio $\sigma_0/m_{gl}$

$$-\vartheta_{gl} \simeq h_{gl}\frac{\sigma_0^2}{m_{gl}^2 - m_p^2} \simeq 10^{-3}\,h_{gl} . \tag{C.21}$$

# References

1. S. Weinberg, Physica **96A** (1979) 327
2. J. Gasser and H. Leutwyler, Phys. Rep. **C87** (1982) 77; Nucl. Phys. **B250** (1985) 465
3. M. Gell–Mann and M. Levy, Nuovo Cim. **16** (1960) 705
4. Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345; D. Gromes, Z. Phys. **C11** (1981) 147; A. Manohar and H. Georgi, Nucl. Phys. **B234** (1984) 189; R. Haymaker, Riv. Nuovo Cim. **14** (1991) No. 8; J. Bijnens, preprint NORDITA 95/10 N,P (hep–ph/9502335); T. Hatsuda and T. Kunihiro, Phys. Rep. **C247** (1994) 221
5. U. Ellwanger and C. Wetterich, Nucl. Phys. **B423** (1994) 137
6. D.–U. Jungnickel and C. Wetterich, Phys. Rev. **D53** (1996) 5142 (hep–ph/9505267)
7. R. D. Pisarski and F. Wilczek, Phys. Rev. **D29** (1984) 338; K. Rajagopal and F. Wilczek, Nucl. Phys. **B399** (1993) 395; Nucl. Phys. **B399** (1993) 577
8. H. Meyer–Ortmanns, H.–J. Pirner and A. Patkos, Phys. Lett. **295B** (1992) 255; Int. J. Mod. Phys. **C3** (1992) 993
9. D. Metzger, H. Meyer–Ortmanns and H.–J. Pirner, Phys. Lett. **321B** (1994) 66; Phys. Lett. **328B** (1994) 547
10. H. Meyer–Ortmanns and B.–J. Schäfer, Phys. Rev. **D53** (1996) 6586
11. H. Goldberg, Phys. Lett. **131B** (1983) 133
12. L.–H. Chan and R.W. Haymaker, Phys. Rev. **D7** (1973) 402
13. C.T. Hill and L. Martinelli, unpublished
14. R.D. Pisarski, preprint hep–ph/9503330
15. For a discussion of the identification of the $a_0$–mesons see The Particle Data Group, Phys. Rev. **D50** (1994) 1146
16. C. Wetterich, Heidelberg preprint HD–THEP–96–2 (hep–th/9602039)
17. N. Tetradis and C. Wetterich, Int. J. Mod. Phys. **A9** (1994) 4029
18. S. Okubo, Prog. Theor. Phys. **27** (1962) 949; M. Gell–Mann and Y. Ne'eman, The Eightfold Way, W.A. Benjamin, New York (1964)
19. The Particle Date Group, Phys. Rev. **D50** (1994)
20. F.F. Donoghue, B.R. Holstein and Y.–C.R. Lin, Phys. Rev. Lett. **55** (1985) 2766
21. V.G. Meissner, Phys. Rep. **161** (1988) 213; G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. **B321** (1989) 311; I. Zahed and G.E. Brown, Phys. Rep. **C142** (1986) 1; M.C. Birse, preprint hep–ph/9603251